An Introduction to Quantum Field Theory

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Contents

1 Introduction

Our aim is to study the dynamics of elementary **particles**. To do that we shall try to construct a theory which is consistent with the basic postulates of quantum mechanics:

- Physical states are represented by vectors in Hilbert space, $|A\rangle$.
- If a system is in state $|A\rangle$ then the probability of observing the system in state $|B\rangle$ is $|\langle B|A\rangle|^2$.
- Observables are represented by hermitian operators, $\hat{O}$.
- Free particles are described by plane waves, i.e. $\psi(x) = \langle x|p\rangle = Ne^{-ip\cdot x}$ where $|\psi(x)|^2 d^3\mathbf{x}$ is the probability of finding a particle of momentum $\mathbf{p}$ in the volume element $\mathbf{x} \to \mathbf{x} + d^3\mathbf{x}$.

We'll also make sure that our theory is consistent with the special theory of relativity.

Note that the postulates above imply that $\hat{p} = -i\nabla$ and $E = i\partial/\partial t$. From which it follows that $[\hat{x}, \hat{p}] = i$ and the Einstein mass-energy relation implies that

$$(\partial_\mu \partial^\mu + m^2)\psi = 0. \tag{1.1}$$

I write $\partial_\mu = (\partial/\partial t, \nabla)$ and use natural units in which $\hbar = c = 1$. Undergraduate quantum mechanics is typically concerned with computing the response of the wavefunction $\psi(x)$ to some external potential, $V(x)$. For example, in the non-relativistic limit one attempts to solve the Schrödinger equation:

$$-\frac{1}{2m}\nabla^2\psi + V\psi = i\frac{\partial\psi}{\partial t}. \tag{1.2}$$

The wavefunction approach is not a bad approximation in many problems but it will not do for us because relativity allows pair creation, i.e. real particle-antiparticle pairs can be created given sufficient energy. In fact, even if the energy is insufficient to create real particles we still need to consider the effect of virtual particles which, according to the uncertainty principle, can exist for a fleeting instant $\delta t \sim 1/\delta E$. Consequently our theory will be a **multiparticle** theory.

2 Particle States

For now let's work with one type of particle only. We'll also assume that particle states can be completely specified by giving the momenta of the particles (i.e. no spin). If there are interactions between the particles then we'll need to specify the time too. For example, a complete set of Heisenberg states is

$\|0; t_0\rangle$	is the state with zero particles at time t_0
$\|\mathbf{p}; t_0\rangle$	is the state with a particle of momentum $\mathbf{p}$ at time t_0
$\|\mathbf{p}, \mathbf{q}; t_0\rangle$	is the state with two particles of momentum $\mathbf{p}$ and $\mathbf{q}$ at time t_0
$\cdots$	

Any state can be written as a linear superposition of these basis states. It is clear that $\langle \mathbf{p}|\,\mathbf{q}\rangle \propto \delta^3(\mathbf{p}-\mathbf{q})$ and we are free to choose the normalisation. I'm going to use

$$\begin{aligned} \langle \mathbf{p}|\,\mathbf{q}\rangle &= (2\pi)^3\delta^3(\mathbf{p}-\mathbf{q}) \\ \langle 0\,|0\rangle &= 1. \end{aligned} \tag{2.1}$$

Note that if I don't label a state with anything other than momenta then you can assume that its a time independent (Heisenberg) state defined at some time which should not need specifying explicitly.

▷ **Exercise 2.1**
Show that our normalisation choice is not Lorentz invariant and that

$$\langle \mathbf{p}|\,\mathbf{q}\rangle = 2E_p\ (2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})$$

is. $(E_p^2 = \mathbf{p}^2 + m^2)$

Before proceeding we need to sort something out. A state of definite momentum is not local in spacetime, so how can we expect to make use of such states in particle physics experiments where particles leave tracks? The key is to realise that the states relevant to particle physics are states of nearly fixed momenta. To see this, let Δx be the position resolution of the particle detector, e.g. $\sim 10^{-6}$m. By the uncertainty principle we can in principle measure the momentum to an accuracy $\Delta p \sim 1/\Delta x \sim 10^{-9}$eV. So we can work with states of definite momenta without introducing a significant error. Put another way, as far as the particle dynamics are concerned the states are infinite plane waves but our detectors see only localised particles.

3 Time Evolution and the S-Matrix

To compute transition probabilities we are going to need to know how systems evolve in time. In the **Schrödinger picture**:

$$i\frac{\partial}{\partial t}\left|A,t\right\rangle_S = H\left|A,t\right\rangle_S \tag{3.1}$$

which has solution

$$\left|A,t\right\rangle_S = e^{-iH(t-t_0)}\left|A,t_0\right\rangle_S . \tag{3.2}$$

All of the time dependence is carried by the state vectors and the operators remain constant. The time translation operator H is often called the Hamiltonian or energy operator.

Since only matrix elements are observable we can choose to trade off time dependence between state vectors and operators. In the **Heisenberg picture** state vectors remain fixed in time. Since

$${}_S\left\langle B,t\right|\hat{O}_S\left|A,t\right\rangle_S = {}_H\left\langle B\right|\hat{O}_H(t)\left|A\right\rangle_H$$

it follows that

$$\hat{O}_H(t) = e^{iH(t-t_0)}\hat{O}_S e^{-iH(t-t_0)} \tag{3.3}$$

and $\left|A,t_0\right\rangle_S = \left|A;t_0\right\rangle_H = \left|A\right\rangle_H$.

We'll be particularly interested in the **interaction picture**. In this picture, the operators carry the same time dependence as the corresponding Heisenberg operators in the non-interacting (free) theory, i.e.

$$\hat{O}_I(t) = e^{iH_0(t-t_0)}\hat{O}_S e^{-iH_0(t-t_0)} \tag{3.4}$$

where H_0 is the energy operator for the theory in the absence of any particle interactions. The state vectors evolve in time too:

$$|A,t\rangle_I = e^{iH_0(t-t_0)}\,|A,t\rangle_S\,. \tag{3.5}$$

▷**Exercise 3.1**
Show that

$$i\frac{\partial \hat{O}_I(t)}{\partial t} = [\hat{O}_I(t), H_0].$$

We'll see later why the interaction picture is so useful but for now let's figure out how states evolve in the interaction picture. We know that

$$|A,t\rangle_I = U(t,t_0)\,|A\rangle_H \tag{3.6}$$

where

$$U(t,t_0) = e^{iH_0(t-t_0)}e^{-iH(t-t_0)} \tag{3.7}$$

is the time evolution operator. Although $H_0 = H_0^I$, H is still a Schrödinger operator. We can get something depending only on interaction picture operators if we consider

$$\begin{aligned} i\frac{\partial U(t,t_0)}{\partial t} &= -H_0 e^{iH_0(t-t_0)}e^{-iH(t-t_0)} + e^{iH_0(t-t_0)}e^{-iH(t-t_0)}H \\ &= e^{iH_0(t-t_0)}H_{\rm int}e^{-iH(t-t_0)} \\ &= H^I_{\rm int}(t)U(t,t_0) \end{aligned} \tag{3.8}$$

where $H = H_0 + H_{\rm int}$. The challenge is now to solve for $U(t,t_0)$. Naively, we might guess that

$$U(t,t_0) = \exp\left[-i\int_{t_0}^t dt'\; H^I_{\rm int}(t')\right].$$

This is not quite right, because $H^I_{\rm int}(t)$ is an operator. It takes a bit of algebra to work out the right answer which I'll just quote (it's textbook stuff):

$$\begin{aligned} U(t,t_0) &= 1 + (-i)\int_{t_0}^t dt_1\; H^I_{\rm int}(t_1) + \frac{(-i)^2}{2}\int_{t_0}^t dt_1\int_{t_0}^t dt_2\; T\left\{H^I_{\rm int}(t_1)H^I_{\rm int}(t_2)\right\} + \cdots \\ &= T\left\{\exp\left[-i\int_{t_0}^t dt'\; H^I_{\rm int}(t')\right]\right\}. \end{aligned} \tag{3.9}$$

There are a couple of points I need to explain regarding this result. First, I have introduced the notion of a **time ordered product**:

$$\begin{aligned} T\left\{H^I_{\rm int}(t_1)H^I_{\rm int}(t_2)\right\} &= H^I_{\rm int}(t_1)H^I_{\rm int}(t_2) \text{ if } t_2 < t_1 \\ &= H^I_{\rm int}(t_2)H^I_{\rm int}(t_1) \text{ if } t_1 < t_2. \end{aligned} \tag{3.10}$$

i.e. operators are written from left to right in order of decreasing time. It's easy to see how to generalise to products of more than two operators. Second, the second line of (3.9) is somewhat schematic although it ought to be clear how to affect the time ordering.

We can now go ahead and write transition amplitudes in the interaction picture. We'll be focussing on particle scattering with an initial state at $t_i = -\infty$ making a transition to some final state at $t_f = +\infty$:

$$ {}_I\langle f, t_f | \, U(t_f, t_i) \, | i, t_i \rangle_I \, . \tag{3.11} $$

In this limit we'll assume that the external states can be approximated by particle states of the free theory. This is an approximation. For example, in QCD one often applies this approximation to compute the scattering of quarks but the scattering is really of hadrons into hadrons. As a result, we can write

$$ \begin{aligned} |\{p_1, p_2, \cdots, p_n\}, -\infty\rangle_I &\approx |\{p_1, p_2, \cdots, p_n\}; -\infty\rangle_H^{\text{free}} \text{ and} \\ |\{k_1, k_2, \cdots, k_m\}, +\infty\rangle_I &\approx |\{k_1, k_2, \cdots, k_m\}; +\infty\rangle_H^{\text{free}} \, . \end{aligned} \tag{3.12} $$

Particle states of the free theory don't need a time label, since the particle content doesn't change in time, i.e. we can write

$$ \begin{aligned} |\{p_1, p_2, \cdots, p_n\}; -\infty\rangle_H^{\text{free}} &= |p_1, p_2, \cdots, p_n\rangle \text{ and} \\ |\{k_1, k_2, \cdots, k_m\}; +\infty\rangle_H^{\text{free}} &= |k_1, k_2, \cdots, k_m\rangle \, . \end{aligned} \tag{3.13} $$

The states on the right-hand-side are Heisenberg states of the free theory defined at some common time (I am ignoring an inconsequential phase here). The all important transition amplitude for $n \to m$ scattering is thus

$$ \begin{aligned} & {}_I\langle \{k_1, k_2, \cdots, k_m\}, +\infty | \, U(\infty, -\infty) \, |\{p_1, p_2, \cdots, p_n\}, -\infty\rangle_I \\ \approx \; & \langle k_1, k_2, \cdots, k_m | \, U(\infty, -\infty) \, |p_1, p_2, \cdots, p_n\rangle \, . \end{aligned} \tag{3.14} $$

$U(\infty, -\infty)$ is commonly called the S-matrix.

4 Creation and Annihilation Operators

For a non-zero S-matrix we require that

$$ S \, |p_1 p_2 \cdots p_n\rangle \sim |k_1, k_2, \cdots, k_m\rangle $$

where the '$\sim$' means that somewhere in the expansion of the left-hand-side must reside the ket on the right-hand-side. For this to be so S must have the capacity to operate on the initial state ket in such a way that it destroys those incoming particles not found in the final state and creates the outgoing particles not present in the initial state. Quite generally we shall be able to write the S-matrix as a function of **creation** and **annihilation** operators.

The creation operator $a^\dagger(p)$ is defined to be the operator which adds a particle of momentum p to the state on which it acts, i.e.

$$ a^\dagger(p) \, |q_1, q_2, \cdots, q_N\rangle = |p, q_1, q_2, \cdots, q_N\rangle \, . \tag{4.1} $$

It's adjoint $a(p)$ removes a particle of momentum p from the state on which it acts. To see that, consider

$$ \langle q_1' \cdots q_M' | \, a(q) \, | q_1 \cdots q_N \rangle = \langle q_1 \cdots q_N | \, a^\dagger(q) \, | q_1' \cdots q_M' \rangle^* \tag{4.2} $$

There are many other ways to implement the delta functions, e.g. we could fix q_2=q_1' etc. This only affects intermediate steps (which could be in the ellipses anyhow).

$$
\begin{aligned}
&= \langle q_1 \cdots q_N | \, q, q'_1 \cdots q'_M \rangle^* \\
&= \delta_{N,M+1} [(2\pi)^3]^N \big\{ \delta^3(\mathbf{q}_1 - \mathbf{q}) \delta^3(\mathbf{q}_2 - \mathbf{q}'_1) \cdots \delta^3(\mathbf{q}_N - \mathbf{q}'_M) \\
&\quad + \delta^3(\mathbf{q}_1 - \mathbf{q}'_1) \delta^3(\mathbf{q}_2 - \mathbf{q}) \cdots \delta^3(\mathbf{q}_N - \mathbf{q}'_M) \\
&\quad + \cdots \\
&\quad + \delta^3(\mathbf{q}_1 - \mathbf{q}'_1) \delta^3(\mathbf{q}_2 - \mathbf{q}'_2) \cdots \delta^3(\mathbf{q}_N - \mathbf{q}) \big\} \\
&= \delta_{N,M+1} \sum_{i=1}^{N} (2\pi)^3 \delta^3(\mathbf{q} - \mathbf{q}_i) \, \langle q'_1 \cdots q'_M | \, q_1, \cdots q_{i-1}, q_{i+1} \cdots, q_N \rangle .
\end{aligned}
$$

This is true for all $\langle q'_1 \cdots q'_M |$ and so

$$
a(q) \, | q_1 \cdots q_N \rangle = \sum_{i=1}^{N} (2\pi)^3 \delta^3(\mathbf{q} - \mathbf{q}_i) \, | q_1 \cdots q_{i-1}, q_{i+1} \cdots q_N \rangle \, , \tag{4.3}
$$

e.g.

$$
\begin{aligned}
a(q) \, |0\rangle &= 0 \\
a(q) \, |q'\rangle &= (2\pi)^3 \delta^3(\mathbf{q} - \mathbf{q}') \, |0\rangle \, .
\end{aligned} \tag{4.4}
$$

After a bit of algebra one can show the following, important, result:

$$
[a(q'), a^\dagger(q)] = (2\pi)^3 \delta^3(\mathbf{q} - \mathbf{q}'). \tag{4.5}
$$

▷ **Exercise 4.1**
Show that (4.5) is consistent with our normalisation choice $\langle p | \, q \rangle = (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{q})$.

▷ **Exercise 4.2**
Evaluate $a(q')a^\dagger(q) \, |p\rangle$ and $a^\dagger(q)a(q') \, |p\rangle$ and hence show that

$$
[a(q'), a^\dagger(q)] \, |p\rangle = (2\pi)^3 \delta^3(\mathbf{q} - \mathbf{q}') \, |p\rangle \, .
$$

You might like to try to generalise this to any state $|p_1 \cdots p_N\rangle$ and in this way prove the validity of the commutation relation (4.5).

Also, it is fairly straightforward to show that

$$
\int \frac{d^3\mathbf{q}}{(2\pi)^3} E_q \; a^\dagger(q) a(q) \, |q_1 \cdots q_N\rangle = \sum_{i=1}^{N} E_{q_i} \, |q_1 \cdots q_N\rangle \, , \tag{4.6}
$$

i.e. the energy operator of the free theory is

$$
H_0 = \int \frac{d^3\mathbf{q}}{(2\pi)^3} E_q \; a^\dagger(q) a(q) \tag{4.7}
$$

and the number operator is

$$
\int \frac{d^3\mathbf{q}}{(2\pi)^3} a^\dagger(q) a(q). \tag{4.8}
$$

▷ **Exercise 4.3**
Prove (4.6).

5 Quantum Fields and Causality

So, we're going to build the S-matrix out of creation and annihilation operators. We need to figure out what particular functions will be useful to us.

Well, we want our theory to be **causal**. Consider two points x and y separated by a spacelike interval, i.e. $(x-y)^2 < 0$, then it is possible to make precise and independent measurements of observables at x and y. Our assumption of causality is to assume that this is true for all spacelike separations. In quantum mechanics this means that

$$[\hat{O}_1(x), \hat{O}_2(y)] = 0 \tag{5.1}$$

for $(x-y)^2 < 0$ and $\hat{O}_{1,2}$ correspond to some observables.

The simplest building block of our causal theory is the following particular combination of creation and annihilation operators:

$$\phi(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} (a(k)e^{-ik\cdot x} + a^\dagger(k)e^{ik\cdot x}). \tag{5.2}$$

First, let's check that it is causal by computing the commutator

$$\begin{aligned}
[\phi(x), \phi(y)] &= \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} \times \\
&\quad \Big[(a(p)e^{-ip\cdot x} + a^\dagger(p)e^{ip\cdot x}), (a(q)e^{-iq\cdot y} + a^\dagger(q)e^{iq\cdot y})\Big] \\
&= \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} \times \\
&\quad \Big\{[a(p), a^\dagger(q)]e^{-ip\cdot x + iq\cdot y} + [a^\dagger(p), a(q)]e^{ip\cdot x - iq\cdot y}\Big\} \\
&= \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{2E_p} \left(e^{ip\cdot(x-y)} - e^{-ip\cdot(x-y)}\right) \\
&= D(x-y) - D(y-x).
\end{aligned} \tag{5.3}$$

$D(x-y)$ is Lorentz invariant since $d^3\mathbf{p}/E_p$ is Lorentz invariant. Referring to Fig.5.1 we can quite easily see that the commutator vanishes for spacelike separations, i.e. $z^2 < 0$. This is so because the commutator vanishes along the line $z^0 = 0$ since

$$\int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{2E_p} e^{-ip\cdot z} = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{2E_p} e^{ip\cdot z}$$

and consequently it vanishes at all points with $z^2 < 0$ since all points in this region are connected to a point at $z_0 = 0$ by a Lorentz transformation. Referring to Fig.5.1, hyperbola 2 is a typical hyperbola ($z^2 =$ constant) connecting points in the region $z^2 < 0$ related by Lorentz transformations (hyperbola 1 connects points related by Lorentz transformations in the region $z^2 > 0$). Thus $\phi(x)$ is a causal operator. We call it the **field** operator.

The energy operator for the free theory can be written in terms of the field, i.e.

$$\begin{aligned}
H_0 &= \int \frac{d^3\mathbf{k}}{(2\pi)^3} E_k \, a^\dagger(k)a(k) \\
&= \int d^3\mathbf{x} : \left[\frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right] :
\end{aligned} \tag{5.4}$$

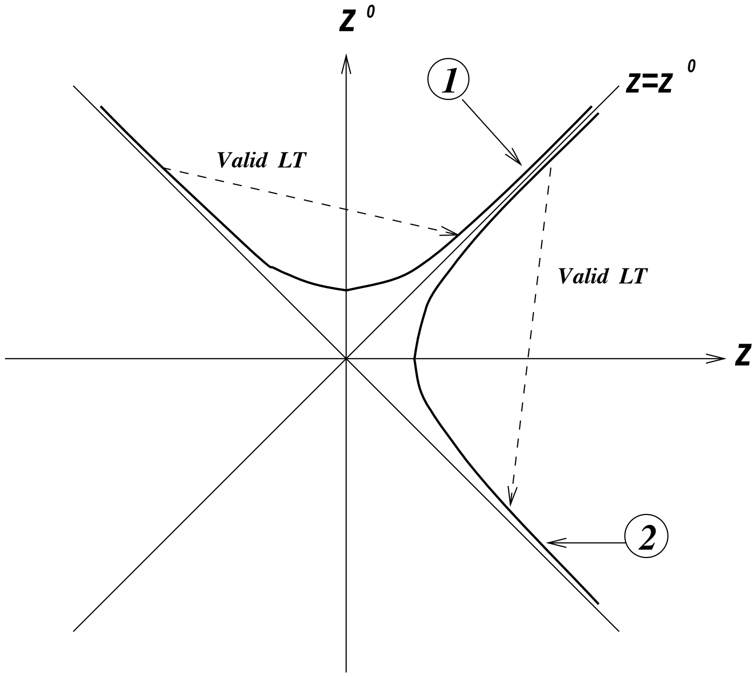


Figure 5.1

where $\Pi = \partial\phi/\partial t$. We've introduced some more new notation. The colons surrounding the integrand tell us to **normal order** the operators enclosed, i.e. they instruct us to put all annihilation operators to the right of all the creation operators. The integrand

$$\mathcal{H} =: \left[\frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right] : \tag{5.5}$$

is called the Hamiltonian density of the free scalar theory.

▷ **Exercise 5.1**
Check (5.5).

While we're busy with these formalities, here's another important result:

$$[\phi(\mathbf{x},t), \Pi(\mathbf{y},t)] = i\delta^3(\mathbf{x}-\mathbf{y}). \tag{5.6}$$

▷ **Exercise 5.2**
Check (5.6).

▷ **Exercise 5.3**
Show that $\phi(x)$ is an interaction picture operator.

$\Pi(x)$ and all the spatial derivatives of $\phi(x)$ are also local interaction picture operators. So, in our quest for the S-matrix we have made some good progress: the interaction Hamiltionian will be a function of the field $\phi(x)$ and its derivatives, i.e.

$$H^I_{\text{int}} = H^I_{\text{int}}(\phi, \nabla\phi, \Pi, \cdots).$$

To proceed we must specify H_{int} (from now on I'm going to drop the 'I' superscript since henceforth all operators will be in the interaction picture). What kinds of H_{int} are realised in nature? At first one might suppose that the list is infinite. But in fact for most systems $\mathcal{H}_{\text{int}} = \mu\phi^3 + \lambda\phi^4$ (μ and λ are so-called coupling constants) will do! Why? Suppose our system has a fundamental length scale, $1/\Lambda$, and that we are interested in physics at scales $\gg 1/\Lambda$, then the only operators which we need to keep in the Hamiltonian density are those with coupling constants of non-negative mass dimension. All other operators are important only in describing physics near the fundamental scale. If we keep only the relevant operators, we say that the theory is **renormalisable**. Renormalisable scalar field theories with simple quartic couplings are appropriate for describing phenomena as diverse as binary liquids, superfluids and ferromagnets. The Higgs sector of the Standard Model is also described by a scalar field with a quartic coupling.

▷ **Exercise 5.4**

Show that if we want a scalar theory to be renormalisable then the interaction Hamiltonian density is

$$\mathcal{H}_{\text{int}} = \mu\phi^3 + \lambda\phi^4.$$

Renormalisation is fascinating subject and well worth the time to study in more detail. A good book on the subject which has nothing to do with particle physics is by Cardy [2]. Sadly, I don't have time to discuss it in any more detail.

We are now ready to compute S-matrix elements. But first, there are a couple of asides to be dealt with. The first aside is a short one. Not suprisingly, the single particle states of the free scalar theory can be described by a wavefunction which satisfies the Klein-Gordon equation. To see this note that the field operator itself satisfies the Klein-Gordon equation:

$$\frac{\partial^2\phi}{\partial t^2} - \nabla^2\phi + m^2\phi = 0. \tag{5.7}$$

From which it follows that the wavefunction

$$\psi(x) = \langle 0|\, \phi(x)\, |p\rangle \tag{5.8}$$

also satisfies the Klein-Gordon equation. So we've recovered the familiar single particle quantum mechanics. We can think of the field operator as creating a particle at some point x when it operates on the vacuum, i.e. $\phi(x)\,|0\rangle \sim |x\rangle$.

▷ **Exercise 5.5**

Compute the wavefunction ψ defined by (5.8). How many (non-interacting) particles per unit volume does this normalisation correspond to?

The second aside will take a lot longer to deal with.

6 Canonical Quantisation of the Classical Field

We're now going to take a step back and derive the quantum field theory of the previous section in a different way (the way that is often discussed in textbooks). Let's start by thinking of $\phi(x)$ as a classical field which satisfies the Klein-Gordon equation.

The Klein-Gordon equation is called the 'equation of motion for the field'. It can be 'derived' by specifying something called the **Lagrangian density**, $\mathcal{L}$, of the theory. We're going to take the trouble to outline the derivation of the Klein-Gordon equation from

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2. \tag{6.1}$$

We do this because it is usual and convenient to define a quantum field theory by specifying its $\mathcal{L}$ rather than, e.g. the equations of motion satisfied by the field operators or the energy operator. Note that the energy operator is not Lorentz invariant whereas the Lagrangian density is.

6.1 Lagrangian and Hamiltonian formulation

Consider a system of N particles. The dynamics of this system is completely specified by giving the locations, $q_i(t)$, and velocities, $\dot{q}_i(t)$, at some instant in time and the Lagrangian, $L = T - V$ (T is the kinetic energy of the system and V is its potential energy).

We are to think of L as a function of $2N$ independent variables: $L = L(q_i, \dot{q}_i)$. The dynamics are determined by the N second order differential equations of Lagrange:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i} \tag{6.2}$$

supplemented with the boundary conditions.

These equations of motion are determined by demanding that the classical trajectories, $q_i(t)$, are such that the integral

$$S = \int_{t_i}^{t_f} dt\ L(q_i, \dot{q}_i) \tag{6.3}$$

is minimised. S is called the **action** and this fundamental principle of classical mechanics is called the least action principle. There is a good discussion of this in the Feynman Lectures [3].

▷ **Exercise 6.1**
For a classical harmonic oscillator, $T = \frac{1}{2}m\dot{x}^2$ and $V = \frac{1}{2}m\omega^2 x^2$. Check that Lagrange's equation gives the right equation of motion.

Another completely equivalent way of formulating the dynamics is to use Hamilton's equations. The Hamiltonian is defined to be a function of the q_i and p_i where

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \tag{6.4}$$

is the canonical momentum (beware, it is not always equal to $m\dot{q}_i$) and

$$H(q_i, p_i) = \sum_{i=1}^{N} p_i \dot{q}_i - L. \tag{6.5}$$

Hamilton's equations are the $2N$ first order differential equations:

$$\frac{\partial H}{\partial p_i} = \dot{q}_i \qquad \text{and} \qquad \frac{\partial H}{\partial q_i} = -\dot{p}_i. \tag{6.6}$$

▷ **Exercise 6.2**
Check out Hamilton's equations for the harmonic oscillator of the previous exercise and show that H is the total energy of the oscillator.

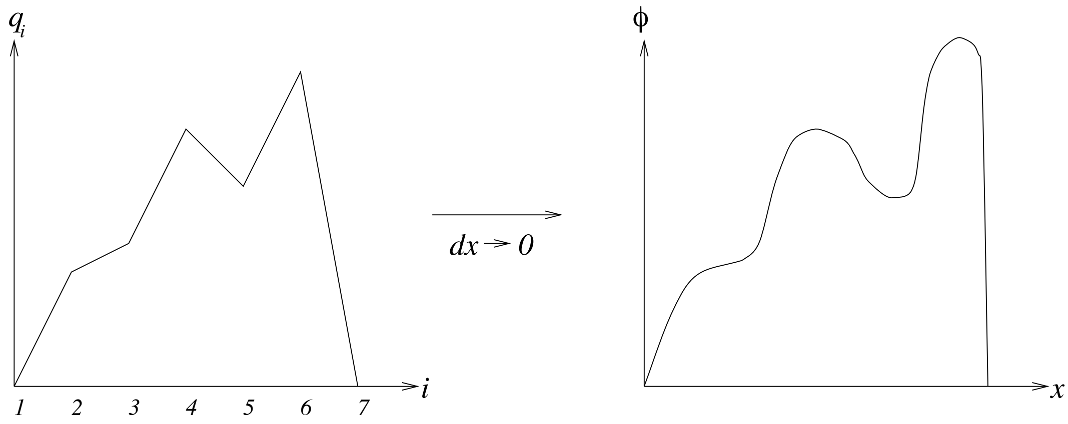


Figure 6.1

The transition to the quantum mechanical system is made by imposing the canonical commutation relations:

$$[q_i, p_j] = i\delta_{ij} \qquad\qquad [q_i, q_j] = [p_i, p_j] = 0. \tag{6.7}$$

The q_i and p_i are now elevated to the status of operators. The field theory of the last section isn't obtained by quantising a classical system of this type. However, it can be obtained by quantising a classical field theory. So let's move on to classical field theory.

For simplicity consider a field in one spatial dimension. We can think of the field as the limit of an N particle system in the limit that the number of particles goes to infinity and the particles become infinitesimally close together, as shown in Fig.6.1. Let's consider a specific field theory, i.e. one with

$$\begin{aligned} T &= \int \frac{1}{2}\rho\ dx\ \left(\frac{\partial\phi(x,t)}{\partial t}\right)^2, \\ V &= \int \left[\frac{1}{2}k\ dx\ \phi(x,t)^2 + \frac{1}{2}\rho\ dx\ \left(\frac{\partial\phi(x,t)}{\partial x}\right)^2\right] \end{aligned} \tag{6.8}$$

Think of ρ at the mass per unit length of the field and k as the spring constant per unit length. The second term in the potential has been added by hand to ensure that the action and Lagrangian density are Lorentz scalars, i.e.

$$S = \int dt\ L = \int dt\ dx\ \mathcal{L} \tag{6.9}$$

where

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\rho\dot{\phi}^2 - \frac{1}{2}k\phi^2 - \frac{1}{2}\rho\left(\frac{\partial\phi}{\partial x}\right)^2 \\ &= \frac{1}{2}\rho(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}k\phi^2. \end{aligned} \tag{6.10}$$

Clearly $\mathcal{L}$ is a Lorentz scalar and so therefore is S (since $dt\ dx \to \gamma dt' dx'/\gamma = dt'\ dx'$ under a boost γ). To make connection with the Klein-Gordon field, pick $\rho = 1$ & $k = m^2$ and generalise to three spatial dimensions.

To minimise the action we need to think of the field and its derivatives as independent variables, i.e.

$$\mathcal{L} = \mathcal{L}(\phi, \dot{\phi}, \partial\phi/\partial x) = \mathcal{L}(\phi, \partial_\mu \phi).$$

People usually assume that higher derivatives are not important. That is an assumption which ultimately is to be tested by data. Avoiding the details, the least action principle leads to the following Euler-Lagrange equations of motion:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = \frac{\partial \mathcal{L}}{\partial \phi}. \tag{6.11}$$

For our relativistic scalar field the Euler-Lagrange equations yield the Klein-Gordon equation.

The canonical momentum conjugate to the field is defined to be

$$\Pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \tag{6.12}$$

and the Hamiltonian density is

$$\mathcal{H}(\phi, \Pi) = \Pi \dot{\phi} - \mathcal{L}. \tag{6.13}$$

Using (6.13) it is easy to show that the Klein-Gordon field is described by the Hamiltonian density

$$\mathcal{H} = \frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2. \tag{6.14}$$

Given the classical field theory, we can go ahead and quantise it by imposing the canonical quantisation conditions. To see how to do this start by imagining our classical field as the limit of a discrete system, i.e.

$$H = \sum_i \delta x \ \left[\Pi_i \dot{\phi}_i - \mathcal{L}_i \right] \tag{6.15}$$

in the limit $\delta x \to 0$ where $q_i = \phi_i$ and $p_i = \Pi_i \delta x$. Thus we impose

$$[\phi_i, \Pi_j] = \frac{i\delta_{ij}}{\delta x}. \tag{6.16}$$

In the limit $\delta x \to 0$, $\phi_i \to \phi(\mathbf{x}_i)$ and $\Pi_j \to \Pi(\mathbf{x}_j)$, i.e.

$$[\phi(\mathbf{x}), \Pi(\mathbf{y})] = i\delta^3(\mathbf{x} - \mathbf{y}). \tag{6.17}$$

If this is not totally clear then it might help to notice that, in the limit of $\delta x \to 0$,

$$\sum_i \delta x \ [\phi_i, \Pi_j] \quad \to \quad \int dx [\phi(\mathbf{x}_i), \Pi(\mathbf{x}_j)] \quad = \quad i.$$

This quantisation is unambiguous in the Schrödinger picture, where the operators are time dependent. It follows that the corresponding relation in the interaction picture is the so-called equal time commutation relation:

$$[\phi(\mathbf{x}, t), \Pi(\mathbf{y}, t)] = i\delta^3(\mathbf{x} - \mathbf{y}). \tag{6.18}$$

and all other commutators vanish. This is exactly (5.6) we found earlier. So, after quantising the classical field whose Lagrangian density is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 \tag{6.19}$$

we arrive at the same multi-particle, relativistic theory of non-interacting scalar particles which we constructed and solved in the previous sections.

Let's summarise. We have learnt a recipe for constructing relativistic theories of particle physics:

(1) Write down a Lagrangian density, $\mathcal{L}(\phi_1, \partial\phi_1, \phi_2, \partial\phi_2, \cdots)$.

(2) Construct the canonical momenta conjugate to the fields.

(3) Impose the equal time commutation relations.

Particle states will 'magically' appear once we interpret the field operators appropriately. I hope I've removed some of the magic by showing in the earlier sections that we are inexorably drawn to quantum field theory in our attempt to construct a relativistic, local, multi-particle theory which is consistent with quantum mechanics.

6.2 Complex Fields and Anti-Particles

What happens if we start from a complex scalar field? For example

$$\mathcal{L} = (\partial_\mu \chi)^\dagger (\partial^\mu \chi) - m_\chi^2 \chi^\dagger \chi \tag{6.20}$$

is the natural generalisation of the real scalar field we discussed previously which is consistent with relativity. The field posseses two independent degrees of freedom and so we think of χ and $\chi^\dagger$ as independent fields:

$$\Pi = \frac{\partial \mathcal{L}}{\partial \dot{\chi}} = \dot{\chi}^\dagger \qquad\qquad \Pi^\dagger = \frac{\partial \mathcal{L}}{\partial \dot{\chi}^\dagger} = \dot{\chi}. \tag{6.21}$$

Quantising the fields means imposing the equal time commutation relations:

$$[\chi(\mathbf{x}, t), \Pi(\mathbf{y}, t)] = [\chi^\dagger(\mathbf{x}, t), \Pi^\dagger(\mathbf{y}, t)] = i\delta^3(\mathbf{x} - \mathbf{y}) \tag{6.22}$$

with all other fundamental commutators equal to zero, e.g.

$$[\chi(\mathbf{x}, t), \chi^\dagger(\mathbf{y}, t)] = 0.$$

We can write down mode expansions for the fields which satisfy the relevant equations of motion:

$$\chi(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} (b(k)e^{-ik\cdot x} + d^\dagger(k)e^{ik\cdot x}). \tag{6.23}$$

Note that since the field is no longer real the Fourier coefficients no longer need be the adjoint of each other. After a bit of algebra you can show that the commutation relations (6.22) imply the important results:

$$\begin{aligned} \left[b(k), b^\dagger(k')\right] &= (2\pi)^3\delta^3(\mathbf{k} - \mathbf{k}') \\ \left[d(k), d^\dagger(k')\right] &= (2\pi)^3\delta^3(\mathbf{k} - \mathbf{k}') \end{aligned} \tag{6.24}$$

and all others vanish. As before the energy operator is constructed from the classical expression with normal ordering of the operators:

$$H = \int \frac{d^3\mathbf{k}}{(2\pi)^3} E_k \left[b^\dagger(k) b(k) + d^\dagger(k) d(k) \right]. \tag{6.25}$$

So, we see that $b^\dagger(k)$ and $d^\dagger(k)$ are independent creation operators. The quanta they create have equal masses (to see this you need to show that the single particle wave-functions $\langle 0 | \chi^\dagger | p \rangle$ and $\langle 0 | \chi | p \rangle$ each satisfy the Klein-Gordon equation with mass m_χ). What distinguishes these quanta?

Well, the single particle wavefunctions satisfy a continuity equation with

$$\rho = i \left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t} \right)$$

and so it should come as no suprise when I tell you that the hermitian operator

$$Q = \int d^3\mathbf{x}\ iq : (\chi^\dagger(x) \Pi^\dagger(x) - \Pi(x) \chi(x)) : \tag{6.26}$$

commutes with the Hamiltonian, $[Q, H] = 0$, i.e. the observable associated with Q is conserved (q is just some arbitrary constant). Furthermore Q can be re-written in terms of creation and annihilation operators:

$$Q = q \int \frac{d^3\mathbf{k}}{(2\pi)^3} \left[b^\dagger(k) b(k) - d^\dagger(k) d(k) \right]. \tag{6.27}$$

The minus sign is critical. It tells us that particles created by $b^\dagger$ carry an amount q of the conserved quantity whilst particles created by $d^\dagger$ carry an amount $-q$. It is usual to call this conserved quantity the **charge**. We'll call the particles carrying charge $+q$ χ^+ bosons and those carrying charge $-q$ χ^- bosons. In all other respects the χ^+ and χ^- bosons are equivalent (there are no more conserved charges). In this way we have introduced the notion of particles and **anti-particles**.

$\triangleright$ **Exercise 6.3**
Check that (6.27) follows from (6.26).

7 Calculating S-Matrix Elements

Let's work with a theory of neutral scalar bosons ϕ which interact with the charged scalars $\chi^\pm$. As our toy model, we'll take

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m_\phi^2 \phi^2 + (\partial_\mu \chi)^\dagger (\partial^\mu \chi) - m_\chi^2 \chi^\dagger \chi - g \chi^\dagger \chi \phi. \tag{7.1}$$

We can read off the interaction Hamiltonian density

$$\mathcal{H}_{\text{int}} = g \chi^\dagger \chi \phi. \tag{7.2}$$

Now that we've specified $\mathcal{H}_{\text{int}}$ we can compute S-matrix elements. Let's start by considering the case of ϕ decay.

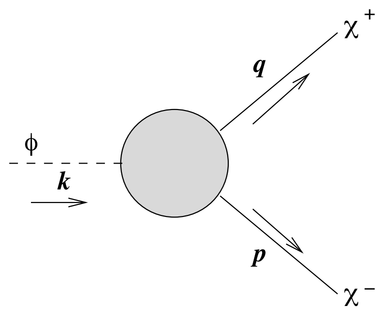


Figure 7.1

7.1 ϕ Decay

Let us deduce the amplitude for $\phi \to \chi^+\chi^-$ shown in Fig.7.1. The initial and final states are

$$\begin{aligned} |k, -\infty\rangle_I &\approx a^\dagger(k)\,|0\rangle \\ |p, q, \infty\rangle_I &\approx b^\dagger(p) d^\dagger(q)\,|0\rangle\,. \end{aligned} \tag{7.3}$$

The S-matrix element, S_{fi}, for this transition is thus

$$S_{fi} = \langle 0|\, b(p) d(q) U(\infty, -\infty) a^\dagger(k)\,|0\rangle \tag{7.4}$$

where

$$\begin{aligned} U(\infty, -\infty) &= T\left\{\exp\left[-i\int dt\ H^I_{\rm int}(t)\right]\right\} \\ &= T\left\{\exp\left[-ig\int d^4x\ \chi^\dagger(x)\chi(x)\phi(x)\right]\right\}. \end{aligned} \tag{7.5}$$

No-one has ever managed to solve analytically using the exact form for U, but if $g \ll 1$ then we can expand the time-ordered exponential in a perturbation series, i.e. neglecting terms $\mathcal{O}(g^2)$ we have

$$U(\infty, -\infty) \approx 1 - ig\int d^4x\ \chi^\dagger(x)\chi(x)\phi(x). \tag{7.6}$$

Time ordering isn't important here since all the fields operate at the same space-time point. Thus

$$\begin{aligned} S_{fi} \approx\ & -ig\int d^4x\ dk'\ dp'\ dq'\ \ \langle 0|\, b(p) d(q) (b^\dagger(p') e^{ip'\cdot x} + d(p') e^{-ip'\cdot x}) \\ & (b(q') e^{-iq'\cdot x} + d^\dagger(q') e^{iq'\cdot x})(a(k') e^{-ik'\cdot x} + a^\dagger(k') e^{ik'\cdot x}) a^\dagger(k)\,|0\rangle \end{aligned} \tag{7.7}$$

and I'm using the space-saving notation

$$dp' = \frac{d^3\mathbf{p}'}{(2\pi)^3}\frac{1}{\sqrt{2E_{p'}}}.$$

The strategy is to use the commutation relations:

$$[a(k), a^\dagger(k')] = (2\pi)^3\delta^3(\mathbf{k} - \mathbf{k}') \qquad \text{etc.} \tag{7.8}$$

to commute all of the annihilation operators to the right (and hence all creation operators to the left), where they vanish on operating upon the vacuum state. To save a lot of tedious writing, it's a good idea to have a think about things first. For example, a little thought should convince you that we only ever need to consider terms with equal numbers of creation and annihilation operators. So,

$$S_{fi} \approx -ig \int d^4x \; dk' \; dp' \; dq' \quad e^{i(p'+q'-k')\cdot x} \langle 0| \, b(p)d(q)b^\dagger(p')d^\dagger(q')a(k')a^\dagger(k) \, |0\rangle \, . \tag{7.9}$$

We can do the x integral using

$$\int d^4x \; e^{i(p'+q'-k')\cdot x} = (2\pi)^4 \delta^4(p'+q'-k'). \tag{7.10}$$

This result ensures the overall conservation of energy and momentum. Focussing on the rest of the integrand:

$$\begin{aligned} \langle 0| \, b(p)b^\dagger(p')d(q)d^\dagger(q')a(k')a^\dagger(k) \, |0\rangle \;\; &= \;\; \langle 0| \, b(p)b^\dagger(p')d(q)d^\dagger(q') \, |0\rangle \, (2\pi)^3\delta^3(\mathbf{k}-\mathbf{k}') \\ &= \langle 0| \, b(p)b^\dagger(p') \, |0\rangle \, (2\pi)^3\delta^3(\mathbf{k}-\mathbf{k}')(2\pi)^3\delta^3(\mathbf{q}-\mathbf{q}') \\ &= \langle 0| \, 0\rangle (2\pi)^9\delta^3(\mathbf{k}-\mathbf{k}')\delta^3(\mathbf{q}-\mathbf{q}')\delta^3(\mathbf{p}-\mathbf{p}'). \end{aligned} \tag{7.11}$$

Putting $\langle 0| \, 0\rangle = 1$ and using the delta functions to do the remaining integrals gives the answer:

$$S_{fi} \approx -ig(2\pi)^4\delta^4(p+q-k)\frac{1}{(2E_k)^{1/2}}\frac{1}{(2E_p)^{1/2}}\frac{1}{(2E_q)^{1/2}}. \tag{7.12}$$

It's usual to define the Lorentz invariant matrix element, $\mathcal{M}_{fi}$:

$$S_{fi} = i\mathcal{M}_{fi}\delta^4(P_f - P_i)\prod_i \frac{1}{(2E_i)^{1/2}} \; \texttt{(2*pi)^4} \tag{7.13}$$

where P_i and P_f are the sum total of the incoming and outgoing four-momenta. For our example

$$\texttt{i}\,\mathcal{M}_{fi} \approx -ig. \tag{7.14}$$

7.2 Propagators and Wick's Theorem

The fact that we didn't have to deal with any time ordering of operators made that last example pretty easy. In general we'll have to deal with time ordered products. For example, the g^2 term in the expansion of $U(\infty, -\infty)$ is

$$(-ig)^2 \int d^4x \int d^4y \; T\left[\chi^\dagger(x)\chi(x)\phi(x)\chi^\dagger(y)\chi(y)\phi(y)\right]. \tag{7.15}$$

Since the ϕ and χ fields commute, the problem generally involves having to work out products like

$$T[\phi(x_1)\phi(x_2)\cdots\phi(x_n)]. \tag{7.16}$$

Let's start with the simplest non-trivial T-product: the product of two fields. We can always re-write it in terms of a normal ordered product plus a function (i.e. not an operator) of the spacetime difference $(x-y)$:[1]

$$T[\phi(x)\phi(y)] =: \phi(x)\phi(y) : +D_F(x-y). \tag{7.17}$$

[1]That the function only depends upon $x-y$ is a consequence of translational invariance.

This is a step in the right direction, because normal ordering is easy to implement in our calculations. All that remains is to compute the function $D_F(x-y)$. Sandwiching the T-product between vacuum states gives

$$\langle 0|\, T[\phi(x)\phi(y)]\, |0\rangle = D_F(x-y). \tag{7.18}$$

$D_F(x-y)$ is called the 'Feynman propagator'. It represents the creation of a particle at some point (either x or y, whichever has the earlier time) and its subsequent propagation to, and destruction at, the other point.

$$\begin{aligned} D_F(x-y) &= \langle 0| \int dk\ dq\ a(k)e^{-ik\cdot x}a^\dagger(q)e^{iq\cdot y}\,|0\rangle\,\Theta(x^0-y^0) \\ &+ \langle 0| \int dk\ dq\ a(q)e^{-iq\cdot y}a^\dagger(k)e^{ik\cdot x}\,|0\rangle\,\Theta(y^0-x^0) \end{aligned} \tag{7.19}$$

where we have introduce the step functions:

$$\begin{aligned} \Theta(x^0-y^0) &= 1 \qquad\qquad x^0 > y^0 \\ &= 0 \qquad\qquad x^0 < y^0. \end{aligned} \tag{7.20}$$

$$\begin{aligned} D_F(x-y) &= \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{2E_k}\Big[e^{-iE_k(x^0-y^0)}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\Theta(x^0-y^0) \\ &+ e^{iE_k(x^0-y^0)}e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\Theta(y^0-x^0)\Big] \\ &= \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon}e^{-ik\cdot(x-y)} \qquad (-\infty < k^0 < \infty). \end{aligned} \tag{7.21}$$

The last expression is very elegant and is explicitly Lorentz invariant (ϵ is an infinitesimally small, real and positive number). I'm not going to go into the mathematics of this last step (it's a straightforward bit of contour integration which you can look up in a text book).

It turns out that a general T-product of fields can be written in terms of normal ordered products and Feynman propagators. The result is known as **Wick's Theorem**:

$$\begin{aligned} T[\phi(x_1)\phi(x_2)\cdots\phi(x_n)] &\equiv T[\phi_1\phi_2\cdots\phi_n] \\ &= \,:\phi_1\phi_2\cdots\phi_n: \\ &+ D_{12}:\phi_3\phi_4\cdots\phi_n: + D_{13}:\phi_2\phi_4\cdots\phi_n: + \cdots \\ &+ D_{12}D_{34}:\phi_5\phi_6\cdots\phi_n: + \cdots \\ &+ \cdots \\ &+ D_{12}D_{34}\cdots D_{n-1,n}\cdots. \end{aligned} \tag{7.22}$$

To keep things compact I've quoted the result for even n and used the notation $D_{12} \equiv D_F(x_1-x_2)$ and $\phi_i \equiv \phi(x_i)$.

7.3 $\chi^+\chi^-$ elastic scattering

We now have all the tools we need to compute S-matrix elements for scattering processes. Let's look at $\chi^+\chi^- \to \chi^+\chi^-$, as shown in Fig.7.2.

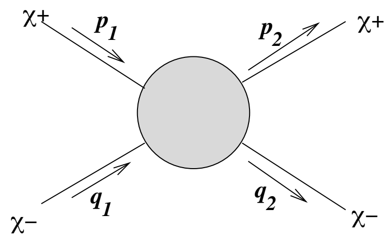


Figure 7.2

The incoming and outgoing states are

$$\begin{aligned}\left|\chi^+\chi^-;-\infty\right\rangle &= b^\dagger(p_1)d^\dagger(q_1)\left|0\right\rangle \\ \left|\chi^+\chi^-;+\infty\right\rangle &= b^\dagger(p_2)d^\dagger(q_2)\left|0\right\rangle \end{aligned}\tag{7.23}$$

and so the required S-matrix element is

$$S_{fi} = \langle 0|\, b(p_2)d(q_2)U(\infty,-\infty)b^\dagger(p_1)d^\dagger(q_1)\,|0\rangle\,. \tag{7.24}$$

To order g^0, $U = 1$ and the particles never interact. We're not interested in that possibility. To order g there is an odd number of creation and annihilation operators sandwiched between the vacuum states, so the matrix element vanishes. The lowest order to give a non-zero contribution is g^2 where we have [Note: notation is sloppy since we throw away terms involving the identity in U.]

$$S_{fi} \approx \langle 0|\, b(p_2)d(q_2)\frac{(-ig)^2}{2}\int d^4y\; d^4z T[\chi^\dagger(y)\chi(y)\phi(y)\chi^\dagger(z)\chi(z)\phi(z)]b^\dagger(p_1)d^\dagger(q_1)\,|0\rangle\,. \tag{7.25}$$

Now we need to use Wick's Theorem to evaluate the T-product. First we must understand how to use Wick's Theorem when there is more than just a single scalar field to consider. You should convince yourself that the only non-vanishing propagators are

$$\begin{aligned} D_F(x-y)|_\phi &= \langle 0|\, T[\phi(x)\phi(y)]\,|0\rangle \\ D_F(x-y)|_\chi &= \langle 0|\, T[\chi(x)\chi^\dagger(y)]\,|0\rangle \\ &= \langle 0|\, T[\chi^\dagger(x)\chi(y)]\,|0\rangle\,. \end{aligned}\tag{7.26}$$

Again, a little thought first reveals that we need only consider terms which involve the Feynman propagator of the ϕ particle (since there are two ϕ fields inside the matrix element but no ϕ fields in either the initial or final states). The T-product is therefore equal to (with $D_F(x-y)|_\chi \equiv D^\chi_{xy}$)

$$\begin{aligned} & D^\phi_{yz}[:\chi^\dagger_y\chi_y\chi^\dagger_z\chi_z: \\ +\;& D^\chi_{yy}:\chi^\dagger_z\chi_z: + D^\chi_{yz}:\chi_y\chi^\dagger_z: \\ +\;& D^\chi_{yz}:\chi^\dagger_y\chi_z: + D^\chi_{zz}:\chi^\dagger_y\chi_y: \\ +\;& D^\chi_{yy}D^\chi_{zz} + D^\chi_{yz}D^\chi_{yz}]. \end{aligned}\tag{7.27}$$

At this stage, we can start to see what is going on. The first term involves

$$D^\phi_{yz}\,\langle 0|\, b(p_2)d(q_2):\chi^\dagger_y\chi_y\chi^\dagger_z\chi_z: b^\dagger(p_1)d^\dagger(q_1)\,|0\rangle\,. \tag{7.28}$$

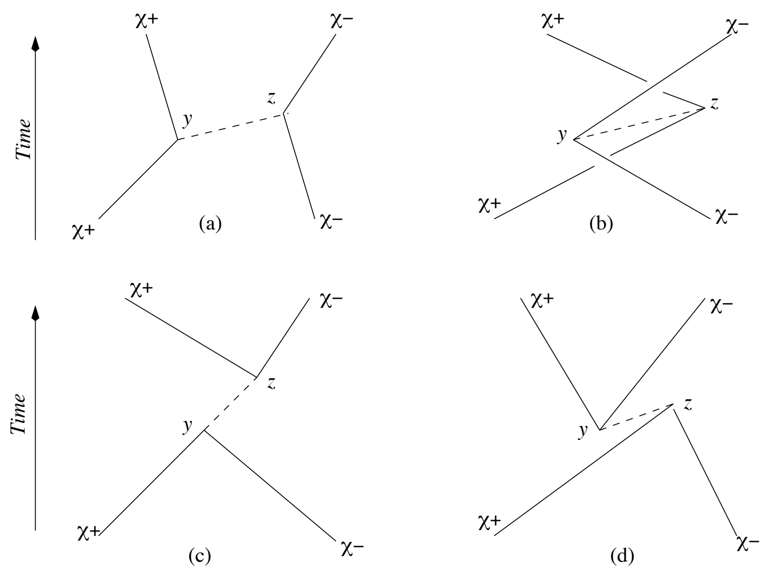


Figure 7.3

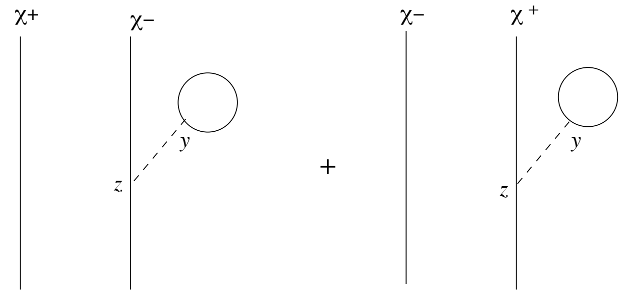


Figure 7.4

The $\chi_y^\dagger$ can either destroy the incoming χ^- or it can create the outgoing χ^+ at the spacetime point y, wheras χ_y can either destroy the incoming χ^+ or create the outgoing χ^- at y. (Take a look back at (6.23) if this is not obvious.) We can represent this first term pictorially. It accounts for the processes shown in Fig.7.3. Since there is a $\chi^\dagger$ and a χ at each point, y and z, it follows that the external particles can be created or destroyed at either y or z. You can see that graphs (a) and (b) must give the same contribution to the S-matrix element after doing the y and z spacetime integrals. Similarly (c) and (d) give equal contributions.

Now let's take a look at the second term:

$$D^\phi_{yz} D^\chi_{yy} \langle 0 | \, b(p_2) d(q_2) : \chi_z^\dagger \chi_z : b^\dagger(p_1) d^\dagger(q_1) \, |0\rangle \, . \tag{7.29}$$

It corresponds to the contributions shown in Fig.7.4. These are not scattering processes: the particle lines are disconnected, which means that the particles don't communicate with each other. Formally speaking, they turned up because I was sloppy about defining the vacuum state in the interacting theory. Avoiding formalities, we employ the rule that we should ignore all those terms which lead to disconnected graphs. All the other terms are of this type, since they involve at least one χ propagator. As an exercise, you should check that they give rise to the graphs shown in Fig.7.5.

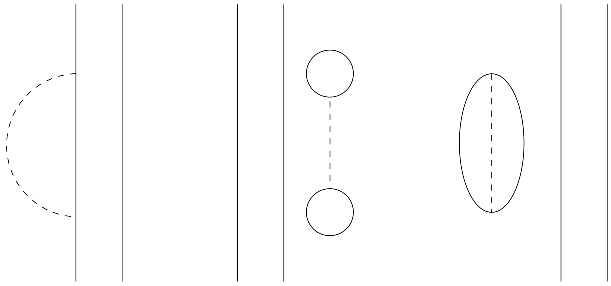


Figure 7.5

By now you should be able to see the connection between the fact that $\mathcal{H}_{\rm int} \sim \chi_x^\dagger \chi_x \phi_x$ and the permissible graphs. Can you?

▷ **Exercise 7.1**
This question is about the order g^3 contribution to the decay $\phi \to \chi^+\chi^-$. The relevant T-product is

$$\left\langle \chi^+\chi^- \right| T[\chi_x^\dagger \chi_x \phi_x \chi_y^\dagger \chi_y \phi_y \chi_y^\dagger \chi_y \phi_y] \left|\phi\right\rangle \tag{7.30}$$

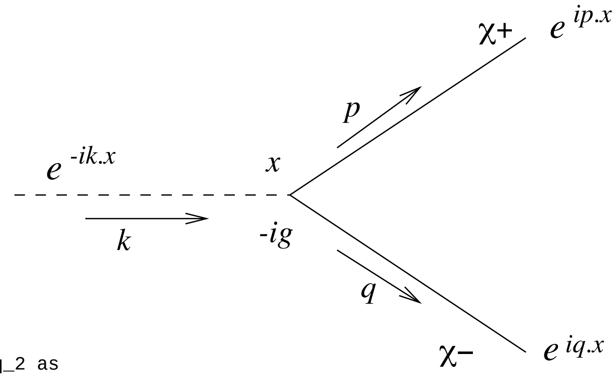
(a) Quickly check that the order g^2 contribution to ϕ decay is zero.
(b) The only non-zero contributions arise from those terms involving one ϕ propagator and two χ propagators. Can you see this?
(c) Use Wick's Theorem to expand the T-product (focus on the relevant terms) and interpret the result diagrammatically.
(d) Check that there are 3! copies of each graph (after integrating over x, y and z) to cancel the 1/3! which sits at the front of the g^3 term in the S-matrix.
Hint: Remember that, at the end of the day, you'll be integrating over all x, y and z. So you can just consider the terms involving (e.g.) the ϕ propagator D^ϕ_{xy}.

Now we can go ahead and compute the S-matrix element of $\chi^+\chi^- \to \chi^+\chi^-$ at 'tree level' (i.e. to order g^2).

$$S_{fi} \approx \frac{(-ig)^2}{2} \int d^4y\ d^4z D_F(y-z)|_\phi \left\langle 0\right| b(p_2)d(q_2) : \chi^\dagger(y)\chi(y)\chi^\dagger(z)\chi(z) : b^\dagger(p_1)d^\dagger(q_1)\left|0\right\rangle . \tag{7.31}$$

The product of the four χ fields is a bit of a mess. We have

$$\begin{aligned}
\approx\ & \frac{(-ig)^2}{2} \int d^4y\ d^4z\ D_F(y-z)|_\phi \int \frac{d^3\mathbf{k_1}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{k}_1}}} \int \frac{d^3\mathbf{k_2}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{k}_2}}} \int \frac{d^3\mathbf{k_3}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{k}_3}}} \\
& \int \frac{d^3\mathbf{k_4}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{k}_4}}} \left\langle 0\right| b_{p_2} d_{q_2} : (b^\dagger_{k_1} e^{ik_1\cdot y} + d_{k_1} e^{-ik_1\cdot y})(b_{k_2} e^{-ik_2\cdot y} + d^\dagger_{k_2} e^{ik_2\cdot y}) \\
& (b^\dagger_{k_3} e^{ik_3\cdot z} + d_{k_3} e^{-ik_3\cdot z})(b_{k_4} e^{-ik_4\cdot z} + d^\dagger_{k_4} e^{ik_4\cdot z}) : b^\dagger_{p_1} d^\dagger_{q_1}\left|0\right\rangle . \\
\approx\ & \frac{(-ig)^2}{2} \int d^4y\ d^4z\ D_F(y-z)|_\phi \int \frac{d^3\mathbf{k_1}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{k}_1}}} \int \frac{d^3\mathbf{k_2}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{k}_2}}} \int \frac{d^3\mathbf{k_3}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{k}_3}}} \\
& \int \frac{d^3\mathbf{k_4}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{k}_4}}} \left\langle 0\right| b_{p_2} d_{q_2} : (b^\dagger_{k_1} b_{k_2} d_{k_3} d^\dagger_{k_4} e^{i(k_1\cdot y - k_2\cdot y - k_3\cdot z + k_4\cdot z)} \\
+\ & b^\dagger_{k_1} b_{k_4} d^\dagger_{k_2} d_{k_3} e^{i(k_1\cdot y + k_2\cdot y - k_3\cdot z - k_4\cdot z)} + b^\dagger_{k_3} b_{k_4} d_{k_1} d^\dagger_{k_2} e^{i(-k_1\cdot y + k_2\cdot y + k_3\cdot z - k_4\cdot z)} \\
+\ & b_{k_2} b^\dagger_{k_3} d_{k_1} d^\dagger_{k_4} e^{i(-k_1\cdot y - k_2\cdot y + k_3\cdot z + k_4\cdot z)}) : b^\dagger_{p_1} d^\dagger_{q_1}\left|0\right\rangle .
\end{aligned} \tag{7.32}$$



Typo in (7.34):
should be p_1, p_2, q_1, q_2 as
the subscripts to the E in the
denominator.

Figure 7.6

Note that we have only retained terms of the form $\sim bb^\dagger dd^\dagger$, since all other combinations give zero. Now let the operators do their job, e.g.

$$\langle 0| \, b_{p_2} d_{q_2} b^\dagger_{k_1} d^\dagger_{k_4} \cdots = \langle 0| \, (2\pi)^3 \delta^3(\mathbf{p}_2 - \mathbf{k}_1)(2\pi)^3 \delta^3(\mathbf{q}_2 - \mathbf{k}_4) \cdots \tag{7.33}$$

$$\begin{aligned} &\approx \frac{(-ig)^2}{2} \int d^4y \; d^4z \; D_F(y-z)|_\phi \frac{1}{\sqrt{2E_{\mathbf{k}_1}}} \frac{1}{\sqrt{2E_{\mathbf{k}_2}}} \frac{1}{\sqrt{2E_{\mathbf{k}_3}}} \frac{1}{\sqrt{2E_{\mathbf{k}_4}}} \langle 0|\, 0 \rangle [e^{i(p_2\cdot y - p_1 \cdot y - q_1 \cdot z + q_2 \cdot z)} \\ &+ \; e^{i(p_2\cdot y + q_2 \cdot y - q_1 \cdot z - p_1 \cdot z)} + e^{i(-q_1\cdot y + q_2 \cdot y + p_2 \cdot z - p_1 \cdot z)} + e^{i(-q_1\cdot y - p_1 \cdot y + p_2 \cdot z + q_2 \cdot z)})]. \end{aligned} \tag{7.34}$$

You should notice that the exponential terms correspond to products of the free particle wavefunctions of the incoming ($\sim e^{-iq\cdot x}$) and outgoing ($\sim e^{iq\cdot x}$) particles. Equation (7.34) has a very physical interpretation which we can easily imagine generalising to other processes. For example, let's revisit the ϕ decay process.

First, draw all possible graphs using only vertices which involve one ϕ particle and two χ particles. For ϕ decay at order g there is just the one graph illustrated in Fig.7.6. Collect together the wavefunctions representing the external (free) particles, a factor $(-ig)$ for each vertex and factors of $(2E)^{-1/2}$ for each external particle and integrate over all possible positions of the vertices. Generally, we'd have to remember to put in any propagator factors too (there are none for the leading order contribution to ϕ decay) and divide by $n!$ for processes at order g^n. Doing this for ϕ decay gives:

$$\begin{aligned} S_{fi} &\approx -ig \int d^4x \frac{e^{ip\cdot x}}{\sqrt{2E_p}} \frac{e^{iq\cdot x}}{\sqrt{2E_q}} \frac{e^{-ik\cdot x}}{\sqrt{2E_k}} \\ &\approx -ig(2\pi)^4 \delta^4(p+q-k) \frac{1}{\sqrt{2E_k}} \frac{1}{\sqrt{2E_p}} \frac{1}{\sqrt{2E_q}} \end{aligned} \tag{7.35}$$

which is just the result we obtained earlier. This physical interpretation of scattering processes in perturbation theory is originally due to Feynman. With these position space Feynman rules we can go ahead and compute S-matrix elements without having to work through a whole pile of tedious algebraic manipulations.

Let's return back to the $\chi^+\chi^-$ elastic scattering process. We can actually press on a little further and do the integrals over y and z since, after putting

$$D_F(y-z)|_\phi = i \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot(y-z)}}{k^2 - m_\phi^2 + i\epsilon} \tag{7.36}$$

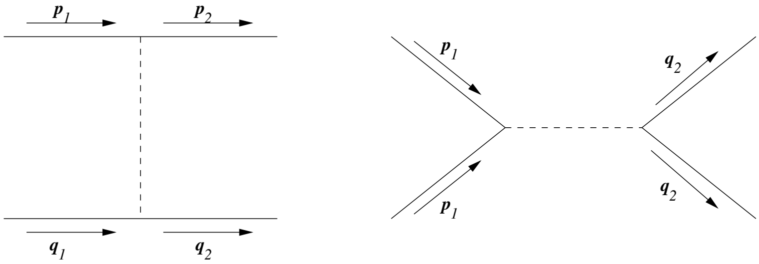


Figure 7.7

Labels in RHS figure are wrong, should be as in LHS figure.

they are just integrals over exponentials (which will lead to delta functions), i.e.

$$
\begin{aligned}
S_{fi} \approx\;& \frac{(-ig)^2}{2}\int d^4k \frac{i}{k^2 - m_\phi^2 + i\epsilon}\int d^4y\; d^4z\; \prod_i \left(\frac{1}{\sqrt{2E_i}}\right)[e^{iy\cdot(p_2-p_1-k)}e^{iz\cdot(q_2-q_1+k)} \\
+\;& e^{iy\cdot(p_2+q_2-k)}e^{iz\cdot(-p_1-q_1+k)} + e^{iy\cdot(q_2-q_1-k)}e^{iz\cdot(p_2-p_1+k)} + e^{iy\cdot(-q_1-p_1-k)}e^{iz\cdot(p_2+q_2+k)}] \\
\approx\;& \frac{(-ig)^2}{2}\int d^4k \frac{i}{k^2 - m_\phi^2 + i\epsilon}[(2\pi)^4\delta^4(p_2-p_1-k)(2\pi)^4\delta^4(q_2-q_1+k) \\
& +(2\pi)^4\delta^4(p_2+q_2-k)(2\pi)^4\delta^4(k-p_1-q_1) \\
& +(2\pi)^4\delta^4(q_2-q_1-k)(2\pi)^4\delta^4(p_2-p_1+k) \\
& +(2\pi)^4\delta^4(k+q_1+p_1)(2\pi)^4\delta^4(k+p_2+q_2)]\prod_i\left(\frac{1}{\sqrt{2E_i}}\right) \\
\approx\;& \frac{(-ig)^2}{2}\left[\frac{2i}{(p_1-p_2)^2 - m_\phi^2} + \frac{2i}{(p_1+q_1)^2 - m_\phi^2}\right] \\
& (2\pi)^4\delta^4(p_1+q_1-p_2-q_2)\prod_i\left(\frac{1}{\sqrt{2E_i}}\right).
\end{aligned} \tag{7.37}
$$

Typo: missing factor of (2*pi)^4 in the denominator.

In the last line we took the limit $\epsilon \to 0$. You can always do the integrals over the spacetime positions of the vertices to move to momentum space where the Feynman rules are even simpler than in position space (the integrals always lead to an overall energy-momentum conserving delta function). To finish off, the Lorentz invariant amplitude is

$$
i\mathcal{M}_{fi} \approx (-ig)^2\left(\frac{i}{t - m_\phi^2} + \frac{i}{s - m_\phi^2}\right). \tag{7.38}
$$

Where $s = (p_1+q_1)^2$ and $t = (p_1-p_2)^2$ are called Mandelstam variables. Diagrammatically, the first term in (7.38) corresponds to the first graph in Fig.7.7, whilst the second term corresponds to the second graph. Notice that the only graphs which we need are those which are topologically distinct.

7.4 Feynman Rules for the $\phi\chi^+\chi^-$ theory

Armed with the Feynman rules, we can go straight to the Lorentz invariant matrix element. One simply draws all topologically distinct Feynman diagrams and employs the following Feynman rules.

- A factor of unity for each external particle. For scalar particles this is a trivial Feynman rule but for particles which carry non-zero angular momentum the wavefunction factors in position space do lead to non-trivial factors which need to be associated with the incoming and outgoing particles, i.e. Dirac spinors for fermions and polarization four-vectors for bosons. These details are discussed in Tim Morris's course.

- For each χ propagator one includes a factor of

$$\frac{i}{p^2 - m_\chi^2 + i\epsilon}$$

where p is the momentum carried by the propagating χ particle.

- For each ϕ propagator one includes a factor of

$$\frac{i}{p^2 - m_\phi^2 + i\epsilon}$$

where p is the momentum carried by the propagating ϕ particle.

- Include a factor of $-ig$ for each $\phi\chi^+\chi^-$ vertex.

- There is an additional rule which is not evident from what we have done so far. One needs to integrate over all momenta not fixed by the external momenta, i.e. for each such 'loop' momentum k one includes a factor

$$\int \frac{d^4k}{(2\pi)^4}.$$

Also, more care is needed to define the contributions to S-matrix elements which arise from diagrams with loops: we'll not deal with loops in this course.

These rules work for all processes, so you don't need to go through all that algebra ever again!

Moreover, you can usually just read off the Feynman rules straight from the Lagrangian density. Recall that, for our scalar theory,

$$\mathcal{L}_{\text{int}} = -g\chi^\dagger\chi\phi. \tag{7.39}$$

This leads directly to the Feynman rule for the vertex. The expressions for the propagators can be obtained from the non-interacting part of the Lagrangian density, which tells us the mode expansion for our interaction picture operators (which is all we need in order to determine the propagators).

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