Lectures on Symmetries in Physics

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   – Feynman rules

• Literature

In order of relevance and difficulty:

1. H.F. Jones: *Groups, Representations and Physics* (IOP, 1998) Second Edition

2. L.H. Ryder, *Quantum Field Theory* (CUP, 1996) Second Edition

3. T.-P. Cheng and L.-F. Li, *Gauge Theory of Elementary Particle Physics* (OUP, 1984).

4. S. Pokorski, *Gauge Field Theories* (CUP, 2000) Second Edition.

5. J. Wess and J. Bagger, *Supersymmetry and Supergravity*, (Princeton University Press, 1992) Second Edition

A list of related problems from H.F. Jones:

1. 2.5, 2.9, 2.12*

2. 3.1, 3.3, 3.4, 3.6

3. 6.1, 6.2, 6.3

4. 9.1

5. 8.1, 8.3, 8.4, 8.5, 8.6, 8.7, 8.8*, 8.9*

6. 10.1, 10.2, 10.3

7. 10.4, 10.5, 10.6, 10.7, 10.8

8. 11.3, 11.5, 11.7, 11.8

Note that more problems as exercises are included in these notes.

1. Introduction to Group Theory

– Definition of a Group G

A *group* $(G, \cdot)$ is a set of elements $\{a, b, c \ldots\}$ endowed with a composition law $\cdot$ that has the following properties:

(i) *Closure*. $\forall\, a, b \in G$, the element $c = a \cdot b \ \in G$.

(ii) *Associativity*. $\forall\, a, b, c \in G$, it holds $a \cdot (b \cdot c) \ = \ (a \cdot b) \cdot c$

(iii) *The identity element* e. $\exists\, e \in G$: $e \cdot a = a, \quad \forall\, a \in G$.

(iv) *The inverse element* a^{-1} *of* a. $\forall\, a \in G, \quad \exists\, a^{-1} \in G$: $a \cdot a^{-1} = a^{-1} \cdot a = e$.

If $a \cdot b = b \cdot a, \ \forall\, a, b \in G$, the group G is called *Abelian*.

– The Discrete Groups S_n, Z_n and C_n

Group G	Multiplication	Order	Remarks
S_n: permutation of n objects	Successive operation	$n!$	Non-Abelian in general
Z_n: integers modulo n	Addition mod n	n	Abelian
C_n: cyclic group $\{e, a, \ldots a^{n-1}\}$ with $a^n = 1$	Unspecified $\cdot$ product	n	$C_n \cong Z_n$

– Cosets and Coset Decomposition

Coset. Let $H = \{h_1, h_2, \dots, h_r\}$ be a *proper* (i.e. $H \neq G$ and $H \neq I = \{e\}$) subgroup of G.
For a given $g \in G$, the sets

$$gH = \{gh_1, gh_2, \dots, gh_r\}\,, \quad Hg = \{h_1 g, h_2 g, \dots, h_r g\}$$

are called the *left* and *right cosets* of H.

Lagrange's Theorem. If $g_1 H$ and $g_2 H$ are two (left) cosets of H, then *either* $g_1 H = g_2 H$ *or* $g_1 H \cap g_2 H = \varnothing$.

Coset Decomposition. If H is a proper subgroup of G, then G can be decomposed into a sum of (left) cosets of H:

$$G \;=\; H \,\cup\, g_1 H \,\cup\, g_2 H \,\cdots\, \cup\, g_{\nu-1} H\,,$$

where $g_{1,2,\dots} \in G$, $g_1 \notin H$; $g_2 \notin H$, $g_2 \notin g_1 H$, etc.

The number ν is called the index of H in G.

The set of all distinct cosets, $\{H, g_1 H, \dots, g_{\nu-1} H\}$, is a manifold, *the coset space*, and is denoted by G/H.

– Normal Subgroup H and Quotient Group G/H

Conjugate to H. If H is a subgroup of G, then the set $H' = gHg^{-1} = \{gh_1 g^{-1}, gh_2 g^{-1}, \dots, gh_r g^{-1}\}$, for a given $g \in G$, is called *g-conjugate* to H or simply *conjugate* to H.

Normal Subgroup H of G. If H is a subgroup of G and $H = gHg^{-1} \;\; \forall\, g \in G$, then H is called a normal subgroup of G.

Groups which contain no proper normal subgroups are termed *simple*.

Groups which contain no proper normal Abelian subgroups are called *semi-simple*.

Quotient Group G/H. Let $G/H = \{H, g_1 H, \dots, g_{\nu-1} H\}$ be the set of all distinct cosets of a normal subgroup H of G, with the multiplication law:

$$(g_i H) \cdot (g_j H) \;=\; (g_i \cdot g_j)\, H\,,$$

where $g_i H, g_j H \in G/H$. Then, it can be shown that $(G/H, \cdot)$ is a group and is termed *quotient group*.

Note that G/H is not a subgroup of G. (*Why?*)

– Morphisms between Groups

Group Homomorphism. If $(A, \cdot)$ and $(B, \star)$ are two groups, then *group homomorphism* is a *functional* mapping f from the set A *into* the set B, i.e. each element of $a \in A$ is mapped into a single element of $b = f(a) \in B$, such that the following multiplication law is preserved:

$$f(a_1 \cdot a_2) = f(a_1) \star f(a_2)\,.$$

In general, $f(A) \neq B$, i.e. $f(A) \subset B$.

Group Isomorphism. Consider a $1:1$ mapping f of $(A, \cdot)$ *onto* $(B, \star)$, such that each element of $a \in A$ is mapped into a single element of $b = f(a) \in B$, and conversely, each element of $b \in B$ is the image resulting from a single element of $a \in A$. If this bijectiv $1:1$ mapping f satisfies the composition law:

$$f(a_1 \cdot a_2) = f(a_1) \star f(a_2)\,,$$

it is said to define an *isomorphism* between the groups A and B, and is denoted by $A \cong B$.

A group homorphism of A into itself is called *endomorphism*.

A group isomorphism of A into itself is called *automorphism*.

2. Group Representations (Reps)

– Definition of a Vector Space V

A vector space V over the field of complex numbers $\mathbb{C}$ is a set of elements $\{\mathbf{v}_i\}$, endowed with two operations $(+, \cdot)$, satisfying the following properties:

(A0) *Closure*. $\mathbf{u} + \mathbf{v} \in V \quad \forall \mathbf{u},\ \mathbf{v} \in V$.

(A1) *Commutativity*. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u} \quad \forall \mathbf{u},\ \mathbf{v} \in V$.

(A2) *Associativity*. $\mathbf{u}+(\mathbf{v}+\mathbf{w}) = (\mathbf{u}+\mathbf{v})+\mathbf{w} \quad \forall \mathbf{u},\ \mathbf{v},\ \mathbf{w} \in V$

(A3) *The identity (null) vector*. $\exists \mathbf{0} \in V$, such that $\mathbf{v} + \mathbf{0} = \mathbf{v}\,, \quad \forall \mathbf{v} \in V$.

(A4) *Existence of inverse*. $\forall \mathbf{v} \in V$, $\exists (-\mathbf{v}) \in V$, such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.

(B0) $\lambda \cdot \mathbf{u} \in V \quad \forall \lambda \in \mathbb{C}, \forall \mathbf{u} \in V$.

(B1) $\lambda \cdot (\mathbf{u} + \mathbf{v}) = \lambda \cdot \mathbf{u} + \lambda \cdot \mathbf{v}$.

(B2) $(\lambda_1 + \lambda_2) \cdot \mathbf{u} = \lambda_1 \cdot \mathbf{u} + \lambda_2 \cdot \mathbf{u}$.

(B3) $\lambda_1 \cdot (\lambda_2 \cdot \mathbf{u}) = (\lambda_1 \lambda_2) \cdot \mathbf{u}$.

(B4) $1 \cdot \mathbf{u} = \mathbf{u}$.

– Definition of a Group Rep.

Group Rep. A *group representation* T,

$$T: \; g \;\rightarrow\; T(g) \;\in\; \mathrm{GL}\,(N, \mathbb{C}) \quad \forall g \in G\,,$$

is a *homomorphism* of the elements g of a group $(G, \cdot)$ into the group GL$(N, \mathbb{C})$ of *non-singular linear* tranformations of a vector space V of dimension N, i.e. the set of $N \times N$-dimensional *invertible* matrices in $\mathbb{C}$.
In addition, *homomorphism* implies that the group multiplication is preserved:

$$T(g_1 \cdot g_2) \;=\; T(g_1)\, T(g_2)\,.$$

...

Two reps. T_1 and T_2 are *equivalent* if there exists an isomorphism ($1:1$ correspondance) between T_1 and T_2. Such an equivalence is denoted as $T_1 \cong T_2$, or $T_1 \sim T_2$.

Two *equivalent* reps may be related by a similarity trans. S: $T_1(g) \;=\; S T_2(g) S^{-1}$ $\forall g \in G$ and S independent of g.

...

Character χ of a rep T of a group G is defined as the set of all traces of the matrices $T(g)$: $\chi = \{\chi(g)/\chi(g) = \sum_i [T(g)]_{ii} \;\wedge\; g \in G\}$.

Corollary: Equivalent reps have the *same* character. Conversely, if two reps have the same character, they are equivalent.

– Reducible and Irreducible Reps.

Reducible rep. A group rep. $T(g)$ is said to be (completely) *reducible*, if there exists a non-singular matrix $M \in \mathrm{GL}\,(N, \mathbb{C})$ independent of the group elements, such that

$$M\,T(g)\,M^{-1} \;=\; \begin{pmatrix} T_1(g) & 0 & \cdots & 0 \\ 0 & T_2(g) & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & T_r(g) \end{pmatrix} \;\forall g \in G.$$

$T_1(g)$, $T_2(g)$, ..., $T_r(g)$ divide T into reps. of lower dimensions, i.e. $\dim(T) = \sum_{i=1}^{r} \dim(T_i)$, and is denoted by the *direct sum*:

$$T(g) \;=\; T_1(g) \oplus T_2(g) \oplus \cdots \oplus T_r(g) \;=\; \sum_{\oplus} T_{(i)}\,.$$

Irreducible rep (Irrep). A group rep. $T(g)$ which *cannot* be written as a direct sum of other reps. is called *irreducible*.

– Direct Products and Clebsch–Gordan Series

Direct Product of Groups. If $(A, \cdot) = (\{a_1, a_2, \ldots, a_n\}, \cdot)$ and $(B, \star) = (\{b_1, b_2, \ldots, b_m\}, \star)$ are two groups with composition laws $\cdot$ and $\star$, respectively,
then a new *direct-product* group $(G, \odot) = (A \times B, \odot)$ can be uniquely defined with elements $g = a \otimes b$. The multiplication law $\odot$ in G is defined as

$$(a_1 \otimes b_1) \odot (a_2 \otimes b_2) \equiv (a_1 \cdot a_2) \otimes (b_1 \star b_2) .$$

Remarks: (i) A and B are normal subgroups of G (*Why?*).
(ii) $A \cong G/B = \{a_1 \otimes B, a_2 \otimes B, \ldots, a_n \otimes B\}$;
$B \cong G/A = \{A \otimes b_1, A \otimes b_2, \ldots, A \otimes b_m\}$.

Direct Product of Irreps. If $D^{(a)}$ and $D^{(b)}$ are two irreps of the group G, a *direct product*, denoted as $D^{(a\times b)}(g_1 g_2) \equiv D^{(a)}(g_1) \otimes D^{(b)}(g_2)$, can be constructed as follows:

$$[D^{(a\times b)}(g_1 g_2)]_{ij;kl} = [D^{(a)}(g_1)]_{ik} \, [D^{(b)}(g_2)]_{jl} .$$

Frequently, direct products of irreps are called *tensor products*.

It can be shown that $D^{(a\times b)}$ is an *irrep* of the (direct) product group $G \times G$.

Clebsch–Gordan Series

If $g_1 = g_2 = g$, then the symmetry of the product group $G \times G$ is reduced to its diagonal G, i.e. $G \times G \to G$.

In this case, $D^{(a)}(g) \otimes D^{(b)}(g)$ may not be an irrep and can be further decomposed into a direct sum of irreps of G:

$$D^{(a)}(g) \otimes D^{(b)}(g) = \sum_{\oplus} a_c \, D^{(c)}(g) .$$

Such a series decomposition is called a *Clebsch–Gordan* series, and the coefficients a_c are the so-called *Clebsch–Gordan* coefficients.

Applications to reps of the continuous groups SO(2), SU(2) and SU(N) will be discussed in the next lectures.

3. Continuous Groups

– SL$(N,\mathbb{C})$; SO(N); SU(N); SO(N,M)

Group	Properties	No. of indep. parameters	Remarks
GL$(N,\mathbb{C})$	det$M \neq 0$	$2N^2$	General rep
SL$(N,\mathbb{C})$	det$M = 1$	$2(N^2-1)$	SL$(N,\mathbb{C})$ $\subset$ GL$(N,\mathbb{C})$
O$(N,\mathbb{R})$	$\sum_{i=1}^N (x^i)^2 = \sum_{i=1}^N (x'^i)^2$	$\frac{1}{2}N(N-1)$	$O^T = O^{-1}$
SO$(N,\mathbb{R})$	as above + det$O = 1$	$\frac{1}{2}N(N-1)$	as above
SU(N)	$\sum_{i=1}^N \lvert x^i\rvert^2 = \sum_{i=1}^N \lvert x'^i\rvert^2$ det$U = 1$	N^2-1	$U^\dagger = U^{-1}$
SO(N,M)	$\sum_{i,j=1}^{N+M} x^i g_{ij} x^j = \sum_{i,j=1}^{N+M} x'^i g_{ij} x'^j$ $g_{ij} = \text{diag}\,(\underbrace{1,\ldots,1}_{N-\text{times}},\ \underbrace{-1,\ldots,-1}_{M-\text{times}})$	?	$\Lambda^T g \Lambda = g$ det $\Lambda = 1$

– Useful Matrix Relations in GL$(N,\mathbb{C})$

Definitions:

$$\text{(i)}\ e^M \equiv \sum_{n=0}^{\infty} \frac{M^n}{n!};$$

$$\text{(ii)}\ \ln M \equiv \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(M-\mathbf{1})^n}{n} = \int_0^1 du\,(M-\mathbf{1})\,[u(M-\mathbf{1})+\mathbf{1}]^{-1},$$

where $M \in$ GL$(N,\mathbb{C})$, i.e. $\det M \neq 0$.

Basic properties: If $[M_1,\ M_2] = 0$ and $M_{1,2} \in$ GL$(N,\mathbb{C})$, then the following relations hold:

$$\text{(i)}\ e^{M_1}\,e^{M_2} = e^{M_1+M_2}, \quad \text{(ii)}\ \ln(M_1 M_2) = \ln M_1 + \ln M_2 .$$

Useful identity:

$$\ln(\det M) = \text{Tr}\,(\ln M).$$

This identity can be proved more easily if M can be diagonalized through a similarity trans: $S^{-1}MS = \widehat{M}$, where $\widehat{M}$ is a diagonal matrix, and noticing that $\ln M = S \ln \widehat{M} S^{-1}$. (*Question*: How?)

– Generators and Exponential rep of Groups
[Examples: SO(2), U(1), SO(3), SU(2)]

SO(2): Transf. of a point $P(x, y)$ under a rotation through ϕ about z axis:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}}_{\equiv O(\phi)} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Note that $O^T(\phi)O(\phi) = \mathbf{1}_2$ and hence $x^2 + y^2 = x'^2 + y'^2$, i.e. $O(\phi)$ is an orthogonal matrix, with det$O$=1.

SO(2) is an Abelian group, since $O(\phi)O(\phi') = O(\phi + \phi') = O(\phi')O(\phi)$.

Taylor expansion of $O(\phi)$ about $\mathbf{1}_2 = O(0)$:

$$O(\delta\phi) = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{:\ \mathbf{1}_2} - i\,\delta\phi \underbrace{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}_{:\ \sigma_2\ =\ i\frac{\partial O(\phi)}{\partial\phi}|_{\phi=0}} + \mathcal{O}[(\delta\phi)^2],$$

with $\sigma_2^2 = \mathbf{1}_2$ and $\sigma_2 = \sigma_2^\dagger$.

Exponential rep for finite ϕ:

$$O(\phi) = \lim_{N\to\infty} [O(\phi/N)]^N = \exp[-i\phi\,\sigma_2].$$

The Pauli matrix σ_2 is the *generator* of the SO(2) group.

U(1): The 2-dim rep of SO(2) in $(V, \mathbb{R})$ can be reduced in $(V, \mathbb{C})$, by means of the trans:

$$M = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix}, \quad M^{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{pmatrix},$$

i.e.

$$M^{-1}\,O(\phi)\,M = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} = D^{(1)}(\phi) \oplus D^{(-1)}(\phi).$$

Both reps, $D^{(1)}(\phi) = e^{i\phi}$ and $D^{(-1)}(\phi) = e^{-i\phi}$, are *faithful* irreps of U(1).

A general irrep of U(1) is

$$D^{(m)}(\phi) = e^{im\,\phi},$$

where $m \in \mathbb{Z}$. (*Question:* What is the generator of U(1)?)

Direct products of U(1)'s:

$$D^{(m)}(\phi) \otimes D^{(n)}(\phi) = D^{(m+n)}(\phi).$$

Spatial rotation of a wave-function:

Unitary operator of rotation of a wave-function:

$$\widehat{U}_R(\delta\phi)\,\psi(r,\theta) \;=\; (1 - i\delta\phi\hat{X})\,\psi(r,\theta) \;=\; \psi(r,\theta-\delta\phi)\,,$$

where

$$\hat{X} = -i\frac{d}{d\theta} \;=\; \frac{\hat{J}_z}{\hbar}$$

is the z-component angular momentum operator.

SO(3): Group of proper rotations in 3-dim about a given unit vector $\mathbf{n} = (n_x, n_y, n_z) = (n_1, n_2, n_3)$, with $\mathbf{n}^2 = 1$.

Rotations about x, y, z-axes:

$$R_1(\phi) \;=\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}\,, \quad R_2(\phi) = \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix}\,,$$

$$R_3(\phi) \;=\; \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}\,.$$

The *generators* $X_i = i\,\frac{dR_i(\phi)}{d\phi}\Big|_{\phi=0}$ of SO(3) are

$$X_1 \;=\; \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}\,, \quad X_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}\,,$$

$$X_3 \;=\; \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,.$$

Equivalently, they can be represented as

$$(X_k)_{ij} \;=\; -i\,\varepsilon_{ijk}\,; \quad \varepsilon_{ijk} \;=\; \begin{cases} 1 & \text{for } (i,j,k) = (1,2,3) \\ & \text{and even permutations,} \\ -1 & \text{for odd permutations,} \\ 0 & \text{otherwise} \end{cases}$$

where ε_{ijk} is the Levi-Civita antisymmetric tensor.

General rep of the Group element of SO(3):

$$R(\phi, \mathbf{n}) \;=\; \exp(-i\phi\,\mathbf{n}\cdot\mathbf{X})\,,$$

with $\mathbf{X} = (X_1, X_2, X_3)$.

Properties of the Generators of SO(3).

Commutation relations:

$$[X_i,\ X_j] \equiv X_iX_j - X_jX_i = i\varepsilon_{ijk}X_k\,.$$

(Need to use that $(X_k)_{ij} = -i\varepsilon_{ijk}$ and $\varepsilon_{ijm}\varepsilon_{klm} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$.)

Jacobi identity:

$$[X_1,\ [X_2,\ X_3]] + [X_3,\ [X_1,\ X_2]] + [X_2,\ [X_3,\ X_1]] = 0\,.$$

...

Irreps of SO(3). These are specified by an *integer* j (the so-called total angular momentum in QM) and are determined by the $(2j+1)\times(2j+1)$-dim rep of the generators $X_i^{(j)}$:

$$[X_3^{(j)}]_{m'm} = \langle jm'|\hat{X}_3|jm\rangle = m\,\delta_{mm'}\,,$$

$$[X_\pm^{(j)}]_{m'm} = \langle jm'|\hat{X}_\pm|jm\rangle = \sqrt{(j\mp m)(j\pm m+1)}\,\delta_{m',m\pm1}\,,$$

with $X_\pm^{(j)} = X_1^{(j)} \pm iX_2^{(j)}$ and $\hat{X}_i = \hat{L}_i/\hbar$.

Exercise: Find the relation between $X_i^{(1)}$ and X_i.

SU(2): Rotation of a *complex* 2-dim vector $\mathbf{v} = (v_1, v_2)$ (with $v_{1,2} \in \mathbb{C}$) through angle θ about $\mathbf{n}$:

$$\mathbf{v}' = U(\theta,\mathbf{n})\,\mathbf{v}\,; \qquad \mathbf{v}^*\cdot\mathbf{v} = \mathbf{v}'^*\cdot\mathbf{v}'\,,$$

with $\det U = 1$ and

$$U(\theta,\mathbf{n}) = \exp(-i\theta\mathbf{n}\cdot\tfrac{1}{2}\boldsymbol{\sigma}) = \cos\tfrac{1}{2}\theta - i\boldsymbol{\sigma}\cdot\mathbf{n}\sin\tfrac{1}{2}\theta\,,$$

where $\mathbf{n}^2 = 1$ and $\boldsymbol{\sigma} = (\sigma_1,\ \sigma_2,\ \sigma_3)$ are the Pauli matrices.

$\therefore\ X_i = \frac{1}{2}\sigma_i$ are the *generators* of SU(2), with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Properties: (i) $\mathrm{Tr}\,\sigma_i = 0$; (ii) $\sigma_i\sigma_j = \delta_{ij}\mathbf{1}_2 + i\,\varepsilon_{ijk}\,\sigma_k$.

Commutation relation: $[X_i,\ X_j] = i\,\varepsilon_{ijk}\,X_k$ i.e. the same *algebra* as of SO(3).

Precise relation between SO(3) and SU(2):

Since $R(0)$ and $R(2\pi)$ [with $R(0) = R(2\pi) = \mathbf{1}_3$] map into different elements $U(0) = \mathbf{1}_2$ and $U(2\pi) = -\mathbf{1}_2$, a *faithful* $1:1$ mapping is

$$\mathrm{SO}(3) \cong \mathrm{SU}(2)/Z_2\,,$$

where $Z_2 = \{\mathbf{1}_2,\ -\mathbf{1}_2\}$ is a normal subgroup of SU(2).

4. Lie Algebra and Lie Groups

–Generators of a Group as Basis Vectors of a Lie Algebra

A Lie algebra L is defined by a set of a number $d(G)$ of *generators* T_a closed under commutation:

$$[T_a,\ T_b]\ =\ T_a\cdot T_b\ -\ T_b\cdot T_a\ =\ if_{ab}^c T_c\,,$$

where f_{ab}^c are the so-called *structure constants* of L.
In addition, the generators T_a's satisfy the Jacobi identity:

$$[T_a,\ [T_b,\ T_c]]\ +\ [T_c,\ [T_a,\ T_b]]\ +\ [T_b,\ [T_c,\ T_a]]\ =\ 0\,.$$

The set T_a of generators define a basis of a $d(G)$-dimensional vector space $(V,\mathbb{C})$.

In the *fundamental rep*, T_a are represented by $d(F)\times d(F)$ matrices, where $d(F)$ is the least number of dimensions needed to generate the continuous group.

Ex: (i) SO(3): $T_a = X_a$; (ii) SU(2): $T_a = \frac{1}{2}\sigma_a$; (iii) U(1): ?

Exponentiation of T_a generates the group elements of the corresponding continuous Lie group:

$$G(\theta,\mathbf{n}) = \exp[-i\theta\mathbf{n}\cdot\mathbf{T}]\ ,$$

with $\mathbf{n}^2 = 1$.

– The Adjoint Representation

The Lie algebra commutator $[T_c,\ \ ]$ (for fixed T_c) defines a linear homomorphic mapping from L to L over $\mathbb{C}$:

$$[T_c,\ \lambda_1 T_a + \lambda_2 T_b]\ =\ \lambda_1[T_c, T_a]\ +\ \lambda_2[T_c,\ T_b]\,,$$

$\forall\ T_a, T_b \subset L$.

For every given $T_a \in L$, $[T_a,\ \ ]$ may be represented in the vector space L by the structure constants themselves:

$$[D_{\mathcal{A}}(T_a)]^c{}_b\ =\ if_{ab}^c\quad (=\ -if_{ba}^c)\,.$$

Such a rep of T_a is called the *adjoint representation*, denoted by $\mathcal{A}$.

The Killing product form is defined as

$$g_{ab}\ \equiv\ (T_a, T_b)_{\mathcal{A}}\ \equiv\ \mathrm{Tr}[D_{\mathcal{A}}(T_a)D_{\mathcal{A}}(T_b)]\quad (\equiv\ \mathrm{Tr}_{\mathcal{A}}(T_aT_b))\,.$$

$g_{ab} = -f_{ac}^d f_{bd}^c$ is called the *Cartan metric*.

The Cartan metric g_{ab} can be used to lower the index of f_{ab}^c:

$$f_{abc}\ =\ f_{ab}^d\, g_{dc}\,.$$

Exercise: Show that $f_{abc}\ =\ -i\,\mathrm{Tr}_{\mathcal{A}}([T_a,\ T_b]\,T_c)$, and that f_{abc} is totally antisymmetric under the permutation of a, b, c: $f_{abc} = -f_{bac} = f_{bca}$ etc.

General Remarks

- If all f^c_{ab}'s are real for a Lie algebra L, then L is said to be a *real* Lie algebra.

- If the Cartan metric g_{ab} is positive definite for a real L, then L is an algebra for a compact group. In this case, g_{ab} can be diagonalized and rescaled to unity, i.e. $g_{ab} = \mathbf{1}_{ab}$. [Ex: the real algebras of SU(N) and SO(N)].

- There is no adjoint representation for Abelian groups. (*Why ?*)

- An *ideal* I is an invariant subalgebra of L, with $[T^I_a,\ T_b] \subset I$, $\forall\ T^I_a \in I$ and $\forall\ T_b \in L$, or symbolically $[I,\ L] \subset I$.

- Ideals I generate normal subgroups of the continuous group generated by L.

- Lie algebras that do not contain any proper ideals are called *simple* (Ex: SO(2), SU(2), SU(3), SU(5), etc).

- Lie algebras that do not contain any proper Abelian ideals are called *semi-simple*. (*Question*: What is the difference between a simple and a semi-simple Lie algebra?)

- A semi-simple Lie algebra can be written as a direct sum of simple Lie algebras: $L = I \oplus P$.

– Normalization of Generators and Casimir operators

The generators of a Lie group $D_R(T_a)$ of a given rep R are normalized as

$$\mathrm{Tr}\,[D_R(T_a)\,D_R(T_b)] \;=\; T_R\,\delta_{ab}\,.$$

For example, in SU(N) [or SO(N)], $T_F = \frac{1}{2}$ for the fundamental rep and $T_{\mathcal{A}} = N$ for the adjoint reps.

Casimir operators $\mathbf{T}^2_R$ of a Lie algebra of a rep R are matrix reps that commute with all generators of L in rep R.

A construction of a Casimir operator $\mathbf{T}^2_R$ in a given rep R of SU(N) [or SO(N)] may be obtained by

$$(\mathbf{T}^2_R)_{ij} = T_{\mathcal{A}} \sum_{a,b=1}^{d(G)} \sum_{k=1}^{d(R)} [D_R(T_a)]_{ik}\, g^{ab}\, [D_R(T_b)]_{kj} \;=\; \delta_{ij}\, C_R\,,$$

where g^{ab} is the inverse Cartan metric satisfying: $g^{ab}\, g_{bc} = \delta^a_c$.

Exercises:

Show that (i) $[\mathbf{T}^2_F,\ T_a] = 0$;

(ii) $T_R\, d(G) \;=\; C_R\, d(R)$;

(iii) $C_F = \frac{N^2-1}{2N}$ and $C_{\mathcal{A}} = N$ in SU(N).

5. Tensors in SU(N)

– Preliminaries

Trans. of a complex vector $\psi_i = (\psi_1, \psi_2, \ldots, \psi_n)$ *in SU(*N*)*:

$$\psi_i \;\to\; \psi'_i = U_{ij}\psi_j \quad (= U_i^{\ j}\psi_j)\,,$$

where $U^\dagger U = UU^\dagger = \mathbf{1}_n$ and $\det U = 1$.

Define the scalar product invariant under SU(N):

$$(\psi, \phi) \;=\; \psi_i^* \phi_i \quad (= \psi^i \phi_i)\,.$$

Hence, the trans. of the c.c. ψ_i^* is

$$\psi_i^* \equiv \psi^i \;\to\; \psi_i'^* = U_{ij}^* \psi_j^* \quad (\text{or } \psi'^i = U^i_{\ j}\psi^j)\,,$$

with $U_i^{\ j} = U_{ij}$, $U^i_{\ j} = U^*_{ij}$ and $U_k^{\ i} U^k_{\ j} = U^i_{\ k} U_j^{\ k} = \delta^i_j$.

...

Higher-rank tensors are defined as those quantities that have the same trans. law as the direct (diagonal) product of vectors:

$$\psi'^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q} \;=\; (U^{i_1}_{\ k_1} U^{i_2}_{\ k_2} \ldots U^{i_p}_{\ k_p})\,(U_{j_1}^{\ l_1} U_{j_2}^{\ l_2} \ldots U_{j_q}^{\ l_q})\,\psi^{k_1 k_2 \ldots k_p}_{l_1 l_2 \ldots l_q}\,.$$

The rank of $\psi^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q}$ is $p+q$, with p contravariant and q covariant indices.

SU(N*) trans. properties of the Kronecker delta* δ^i_j *and Levi-Civita symbol* $\varepsilon^{i_1 i_2 \ldots i_n}$*:*

Invariance of δ^i_j *under an SU(*N*) trans:*

$$\delta'^i_j = U^i_{\ k} U_j^{\ l}\, \delta^k_l \;=\; U^i_{\ k} U_j^{\ k} \;=\; \delta^i_j\,.$$

The Levi-Civita symbol $\varepsilon^{i_1 i_2 \ldots i_n}$:

$$\varepsilon^{i_1 i_2 \ldots i_n} \;=\; \begin{cases} 1 & \text{if } (i_1, \ldots i_n) \text{ is an } \textit{even} \text{ permutation of } (1, \ldots n) \\ -1 & \text{if } (i_1, \ldots i_n) \text{ is an } \textit{odd} \text{ permutation of } (1, \ldots n) \\ 0 & \text{otherwise} \end{cases}$$

Note that $\varepsilon_{i_1 i_2 \ldots i_n}$ is defined to be fully antisymmetric, such that $\varepsilon_{j i_2 \ldots i_n}\, \varepsilon^{i i_2 \ldots i_n} = (n-1)!\; \delta^i_j$.

Invariance of $\varepsilon_{i_1 i_2 \ldots i_n}$ *(and* $\varepsilon^{i_1 i_2 \ldots i_n}$*) under an SU(*$N$*) trans:*

$$\begin{aligned} \varepsilon'_{i_1 i_2 \ldots i_n} &= U_{i_1}^{\ j_1} U_{i_2}^{\ j_2} \ldots U_{i_n}^{\ j_n}\, \varepsilon_{j_1 j_2 \ldots j_n} \\ &= \underbrace{\det U}_{=\,1}\, \varepsilon_{i_1 i_2 \ldots i_n} \;=\; \varepsilon_{i_1 i_2 \ldots i_n}\,. \end{aligned}$$

Reduction of higher-rank tensors:

Lower-rank tensors can be formed by appropriate use of δ^i_j and $\varepsilon^{i_1 i_2 \ldots i_n}$:

$$
\begin{aligned}
\psi^{i_2\ldots i_p}_{j_2\ldots j_q} &= \delta^{j_1}_{i_1}\,\psi^{i_1 i_2\ldots i_p}_{j_1 j_2\ldots j_q}\,,\\
\psi^{i_1} &= \varepsilon^{i_1 i_2\ldots i_n}\psi_{i_2\ldots i_n}\,,\\
\psi &= \varepsilon^{i_1 i_2\ldots i_n}\psi_{i_1 i_2\ldots i_n}\,,\\
\psi^{i_1 j_1} &= \varepsilon^{i_1 i_2\ldots i_n}\,\varepsilon^{j_1 j_2\ldots j_n}\;\psi_{i_2\ldots i_n j_2\ldots j_n}\,.
\end{aligned}
$$

Since the Levi-Civita tensor can be used to lower or raise indices, we only need to study tensors with upper or lower indices.

Exercise: Show that ψ is an SU(N)-invariant scalar.

– Young Tableaux

Higher-rank SU(N) tensors do *not* generally define bases of irreps. To decompose them into irreps, we exploit the following property which is at the heart of Young Tableaux.

An illustrative example. Consider the 2nd rank tensor ψ_{ij}, with the trans. property:

$$\psi'_{ij} \;=\; U_i^{\ k} U_j^{\ l}\,\psi_{kl}\,.$$

Permutation of $i \leftrightarrow j$ (denoted by P_{12}) does not change the trans. law of ψ_{ij}:

$$
\begin{aligned}
P_{12}\,\psi'_{ij} \;=\; \psi'_{ji} \;=\; U_j^{\ k} U_i^{\ l}\,\psi_{kl} &\;=\; U_j^{\ l} U_i^{\ k}\psi_{lk}\\
&\;=\; U_j^{\ l} U_i^{\ k}\;P_{12}\,\psi_{kl}\,.
\end{aligned}
$$

Hence, P_{12} can be used to construct the following irreps:

$$
\begin{aligned}
S_{ij} &= \frac{1}{2}(1+P_{12})\,\psi_{ij} \;=\; \frac{1}{2}\,(\,\psi_{ij}\;+\;\psi_{ji}\,)\,,\\
A_{ij} &= \frac{1}{2}(1-P_{12})\,\psi_{ij} \;=\; \frac{1}{2}\,(\psi_{ij}\;-\;\psi_{ji})\,,
\end{aligned}
$$

with $P_{12}S_{ij} = S_{ij}$ and $P_{12}A_{ij} = -A_{ij}$, since there is no mixing between S_{ij} and A_{ij} under an SU(N) trans:

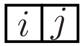
$$S'_{ij} \;=\; U_i^{\ k} U_j^{\ l} S_{kl}\,,\quad A'_{ij} \;=\; U_i^{\ k} U_j^{\ l} A_{kl}\,.$$

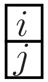
Introduction to Young Tableaux

A complex (covariant) vector (or state) ψ_i in SU(N) is represented by a $\square$:

$\psi_i \equiv$ 

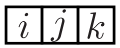
The operation of symmetrization and antisymmetrization is represented as

$\psi_{(ij)} \equiv$ 

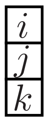
$\psi_{[ij]} \equiv$ 

with $\psi_{(ij)} = \frac{1}{2}(1 + P_{12})\,\psi_{ij} = S_{ij} = S_{ji}$ and
$\psi_{[ij]} = \frac{1}{2}(1 - P_{12})\,\psi_{ij} = A_{ij} = -A_{ji}$.

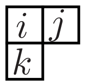
By analogy, for ψ_{ijk} we have



$\psi_{(ijk)}$



$\psi_{[ijk]}$



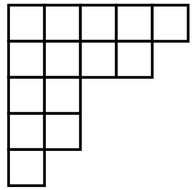
$\psi_{[(ij);k]}$

where $\psi_{(ijk)}$ is fully symmetric in i, j, k,
$\psi_{[ijk]}$ is fully anti-symmetric in i, j, k and
$\psi_{[(ij);k]} = (1 - P_{13})\,(1 + P_{12})\,\psi_{ijk}$.

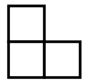
Exercise: Express $\psi_{(ijk)}$ and $\psi_{[(ij);k]}$ in terms of ψ_{ijk}.
(*Ans:* $\psi_{[(ij);k]} = \psi_{ijk} + \psi_{jik} - \psi_{kji} - \psi_{jki}$.)

Rules for constructing a legal Young Tableau

- A typical Young tableau for an (n-rank) tensor with n indices looks like:

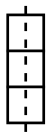


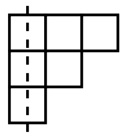
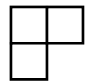
- Each row of a Young tableau must contain no more boxes than the row above. This implies e.g. that



  is not a valid diagram.

- There should be no column with more than N boxes for SU(N). In this respect, a column with exactly N boxes can be crossed out. For example, in SU(3) we have:

 $= \mathbf{1}$

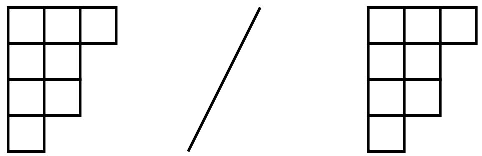
 $=$ 

(*Why?*)

How to find the dimension of a Young Tableau rep

Steps to be followed:

(a) Write down the ratio of two copies of the tableau:



(b) *Numerator*: Start with the number N for SU(N) in the top left box. Each time you meet a box, increase the previous number by $+1$ when moving to the right in a row and decrease it by -1 when going down in a column:

N	$N+1$	$N+2$
$N-1$	N	
$N-2$	$N-1$	
$N-3$		

(c) *Denominator*: In each box, write the number of boxes being to its right + the number being below of it and add $+1$ for itself:

6	4	1
4	2	
3	1	
1		

(d) The dimension d of the rep is the ratio of the products of the entries in the numerator versus that in the denominator:

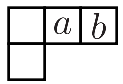
$$d = [N(N+1)(N+2)(N-1)N(N-2)(N-1)(N-3)] \\ / [6 \times 4 \times 4 \times 2 \times 3] \, .$$

Rules for Clebsch-Gordan series

The direct product of reps can be decomposed as a Clebsch-Gordan series (or direct sum) of irreps. This reduction can be performed systematically by means of Young Tableaux, following the rules below:

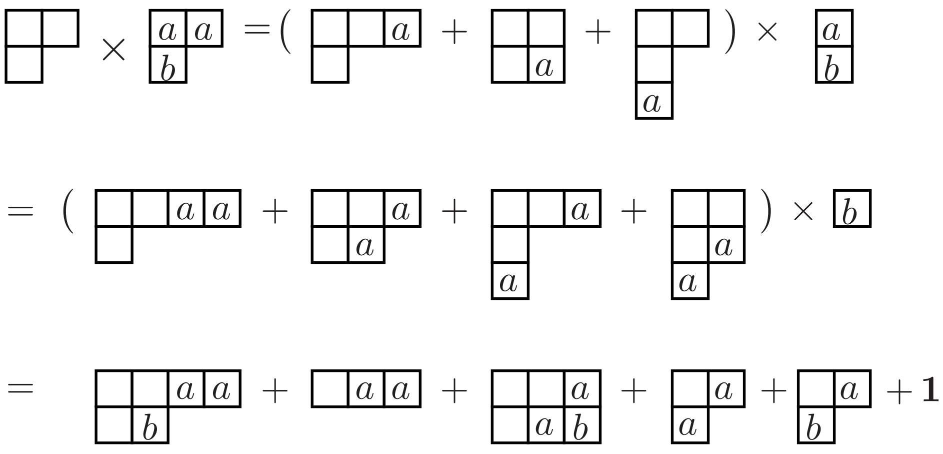
(a) Write down the two tableaux T_1 and T_2 and label successive rows of T_2 with indices $a, b, c, \ldots$:

(b) Attach boxes a, b, c ... from T_2 to T_1 in all possible ways one at a time. The resulting diagram should be a legal Young tableau with no two a's or b's being in the same column (because of cancellation due to antisymmetrization).

(c) At any given box position, there should be no more b's than a's to the right and above of it. Likewise, there should be no more c's than b's etc. For example, the tableau  is not legal.

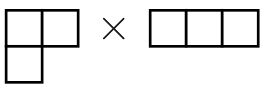
(d) Two generated tableaux with the same shape are different if the labels are distributed differently.

An example in SU(3)



$$8 \times 8 = 27 \oplus 10 \oplus \overline{10} \oplus 8 \oplus 8 \oplus 1$$

Exercise: Find the Clebsch–Gordan decomposition of the product 8×10 in SU(3), represented by Young tableaux as



(*Ans:* $8 \times 10 = 8 \oplus 10 \oplus 27 \oplus 35$)

– Applications to Particle Physics

The SU(3) quark symmetry

Define the quark-basis states

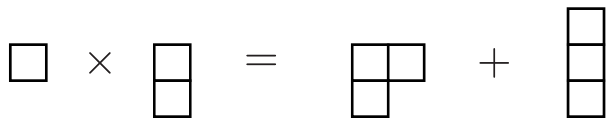
$$q_i = \begin{pmatrix} u \\ d \\ s \end{pmatrix}, \qquad q^i = \begin{pmatrix} \bar{u} \\ \bar{d} \\ \bar{s} \end{pmatrix}.$$

Then, $q_i \equiv 3$ and $q^i \equiv \bar{q}_i \equiv \bar{3}$.

Clebsch–Gordan series: $3 \otimes \bar{3} = 8 \oplus 1$:

$$q_i q^j = (q_i q^j - \frac{1}{3}\delta_i^j \, q_k q^k) + \frac{1}{3}\delta_i^j \, q_k q^k .$$

In terms of Young–Tableaux:



The singlet state is $\eta_1 = \frac{1}{\sqrt{3}} q_i q^i = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$.

The remaining 8 components represent the pseudoscalar octet $P_i^j = (q_i q^j - \frac{1}{3}\delta_i^j \, q_k q^k)$:

$$P_j^i = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta_8 \end{pmatrix}.$$

Baryons as three-quark states:

Clebsch–Gordan series: $3 \otimes 3 \otimes 3 \ = \ 10 \oplus 8 \oplus 8 \oplus 1$

Define $q_{ijk} = q_i q_j q_k$, then

$$q_{ijk} \ = \ q_{(ijk)} \ + \ q_{[(ij);k]} \ + \ q_{[(ji);k]} \ + \ q_{[ijk]} \,.$$

For example, the baryon-octet may be represented by

$$B \ = \ q_{[(ij);k]} \ = \ \begin{pmatrix} \frac{1}{\sqrt{2}}\Sigma^0 + \frac{1}{\sqrt{6}}\Lambda_8 & \Sigma^+ & p \\ \Sigma^- & -\frac{1}{\sqrt{2}}\Sigma^0 + \frac{1}{\sqrt{6}}\Lambda & n \\ \Xi^- & \Xi^0 & -\frac{2}{\sqrt{6}}\Lambda \end{pmatrix} .$$

Exercise: Find the Clebsch–Gordan decomposition
for $3 \otimes 3 \otimes 3$, using Young–Tableaux.
What is the quark wave-function of p and n?

Particle assignment in an SU(5) unified theory

The particle content of the SM = SU(3)$_c$⊗SU(2)$_L$⊗ U(1)$_Y$ consists of three generations of quarks and leptons.

One generation of quarks and leptons in the SM contains 15 dynamical degrees of freedom:

$$\begin{pmatrix} u_L^{r,g,b} \\ d_L^{r,g,b} \end{pmatrix} , \qquad \begin{pmatrix} \nu_L \\ l_L \end{pmatrix} , \qquad u_R^{r,g,b} , \qquad d_R^{r,g,b} , \qquad l_R \,.$$

In SU(5), the SM fermions are assigned as follows:

$$\bar{5} \ = \ \begin{pmatrix} \bar{d}^r \\ \bar{d}^g \\ \bar{d}^b \\ e \\ -\nu \end{pmatrix}_L ,$$

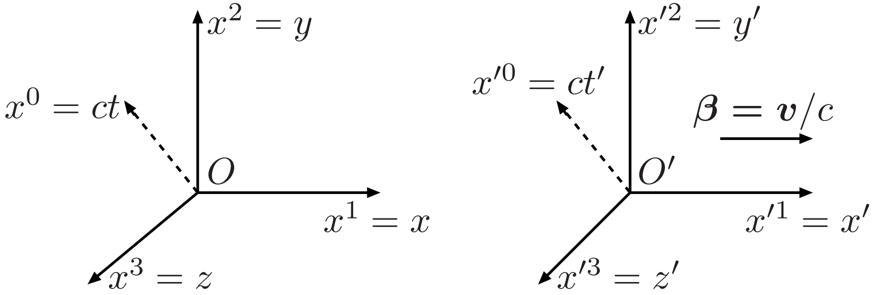
and

$$10 \ = \ \begin{pmatrix} 0 & \bar{u}^b & -\bar{u}^g & u^r & d^r \\ -\bar{u}^b & 0 & \bar{u}^r & u^g & d^g \\ \bar{u}^g & -\bar{u}^r & 0 & u^b & d^b \\ -u^r & -u^g & -u^b & 0 & \bar{e} \\ -d^r & -d^g & -d^b & -\bar{e} & 0 \end{pmatrix}_L$$

Exercise: Given that $\bar{5}$ is the complex conjugate rep $\psi_i^* = \psi^i$ of the SU(5) in the fundamental rep, find the tensor rep for the 10-plet representing the remaining fermions of the SM.

6. Lorentz and Poincaré Groups

Lorentz trans:



$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}(\boldsymbol{\beta})\, x^{\nu},$$

where $x^{\mu} = (ct, x, y, z)$, $x'^{\mu} = (ct', x', y', z')$ are the contravariant position 4-vectors, and

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{for } \boldsymbol{\beta} \parallel \mathbf{e}_x .$$

Given the metric $g_{\mu\nu} = \mathrm{diag}\,(1, -1, -1, -1)$, the covariant 4-vector is defined as $x_{\mu} = g_{\mu\nu}\, x^{\nu} = (ct, -x, -y, -z)$.

Under a Lorentz trans, we have $x^{\mu}x_{\mu} = x'^{\mu}\, x'_{\mu}$ or

$$x^{\mu}\, g_{\mu\nu} x^{\nu} = x^{\beta} \Lambda^{\mu}{}_{\beta} g_{\mu\nu} \Lambda^{\nu}{}_{\alpha} x^{\alpha} \Rightarrow \Lambda^T g \Lambda = g,$$

so $\Lambda^{\mu}{}_{\nu} \in$ SO(1,3), with $\det \Lambda = 1$.

– Lie Algebra and Generators of the Lorentz Group

Generators and Lie Algebra of SO(1,3)

Generators of rotations $J_{1,2,3}$:

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix},$$

$$J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Generators of boosts $K_{1,2,3}$:

$$K_1 = \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$K_3 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}.$$

Commutation relations of the Lie algebra SO(1,3):

$$\begin{aligned} [J_i,\ J_j] &= i\,\varepsilon_{ijk}\, J_k\,, \\ [J_i,\ K_j] &= i\,\varepsilon_{ijk}\, K_k\,, \\ [K_i,\ K_j] &= -i\,\varepsilon_{ijk}\, J_k \end{aligned}$$

SO(1,3)$_\mathbb{C}$ $\cong$ SU(2) $\times$ SU(2) [or SO(1,3)$_\mathbb{R}$ $\sim$ SL(2,$\mathbb{C}$)]

Define

$$\mathbf{X}^{\pm} \;=\; \frac{1}{2}(\mathbf{J} \pm i\,\mathbf{K})\,,$$

then

$$\begin{aligned} [X_i^+,\; X_j^+] &= i\,\varepsilon_{ijk}\,X_k^+\,, \\ [X_i^-,\; X_j^-] &= i\,\varepsilon_{ijk}\,X_k^-\,, \\ [X_i^+,\; X_j^-] &= 0\,. \end{aligned}$$

Hence, SO(1,3) algebra splits into two SU(2) ones:

$$\mathrm{SO}(1,3)_\mathbb{C} \cong \mathrm{SU}(2) \times \mathrm{SU}(2)\,,$$

where SO(1,3)$_\mathbb{C}$ is the rep from a complexified SO(1,3) algebra. However, there is an 1:1 correspondence of the reps between SO(1,3)$_\mathbb{C}$ and SO(1,3)$_\mathbb{R}$. In fact, we have the homomorphism

$$\mathrm{SO}(1,3)_\mathbb{R} \;\sim\; \mathrm{SL}(2,\mathbb{C})\,,$$

which is more difficult to use for classification of reps.

Classification of basis-states reps in SO(1,3)

We enumerate basis-state reps in SO(1,3) by (j_1, j_2), using the relation of SO(1,3) with $\mathrm{SU}(2)_1 \times \mathrm{SU}(2)_2$, where $j_{1,2}$ are the total spin numbers with respect to $\mathrm{SU}(2)_{1,2}$. The total degrees of freedom are $(2j_1+1)(2j_2+1)$. In detail, we have

(0,0): This is a total spin zero rep, with dim one. $(0,0)$ represents a scalar field $\phi(x)$ satisfying the Klein-Gordon equation: $(\Box + m^2)\phi(x) = 0$, where $\Box = \partial^\mu\partial_\mu$.

$(\frac{1}{2}, 0)$: This a 2-dim rep, the so-called *left-handed Weyl rep*, e.g. neutrinos. It is denoted with a 2-dim complex vector ξ_α, usually called the left-handed Weyl spinor. Under a Lorentz trans, ξ_α transforms as

$$\xi'_\alpha = M_\alpha^{\;\beta}\,\xi_\beta\,,$$

where $M_\alpha^{\;\beta} \in \mathrm{SL}(2,\mathbb{C})$.

$(0, \frac{1}{2})$: This is the corresponding 2-dim rep of the *right-handed Weyl spinor* and is denoted as $\bar\eta_{\dot\alpha}$, which transforms under Lorentz trans as

$$\bar\eta'_{\dot\alpha} = M^{\dagger\,\dot\beta}_{\;\;\dot\alpha}\,\bar\eta_{\dot\beta}\,,$$

where $M^{\dagger\,\dot\beta}_{\;\;\dot\alpha} \in \mathrm{SL}(2,\mathbb{C})$.

$(\frac{1}{2}, \frac{1}{2})$: This is the defining 4-dim rep, describing a spin 1 particle with 4 components. One can use the matrix rep: $A^\mu(\sigma_\mu)_{\alpha\dot\alpha}$ or simply A^μ, e.g. $A^\mu = (\Phi/c,\; \mathbf{A})$ in electromagnetism.

– Lie Algebra and Generators of the Poincaré Group

The Poincaré trans consist of Lorentz trans plus space-time translations:

$$x'^{\mu} \;=\; \Lambda^{\mu}{}_{\nu}\, x^{\nu} \;+\; a^{\mu}\,,$$

where a^{μ} is a constant 4-vector.

The generator of translations in a differential-operator rep is

$$P^{\mu} \;=\; i\partial^{\mu} \;=\; i\frac{\partial}{\partial x_{\mu}} \;=\; i\,(\,\frac{\partial}{c\partial t}\,,\; -\nabla\,)\,,$$

with $P_{\mu} = i\partial_{\mu} = i(\frac{\partial}{c\partial t}\,,\ \nabla)$, because

$$e^{-ia^{\nu}P_{\nu}}x^{\mu} \;=\; x^{\mu} + a^{\mu}\,.\ (\textit{Why?})$$

An analogous differential-operator rep of the 6-generators of Lorentz trans is given by the *generalized angular momentum* operators:

$$L_{\mu\nu} \;=\; x_{\mu}P_{\nu} \;-\; x_{\nu}P_{\mu}\,,$$

with the identification

$$J_i \;=\; \frac{1}{2}\varepsilon_{ijk}L_{jk}\,, \qquad K_i \;=\; L_{0i}\,.$$

Exercise: Show that $J_i \;=\; \frac{1}{2}\,\varepsilon_{ijk}L_{jk}$ and $K_i = L_{0i}$ satisfy the SO(1,3) algebra.

The Lie Algebra of the Poincaré Group:

The commutation relations defining the Poincaré Lie algebra are

$$\begin{aligned}
[P_{\mu}\,,\ P_{\nu}] &= 0\,,\\
[P_{\mu}, L_{\rho\sigma}] &= i\,(g_{\mu\rho}P_{\sigma} - g_{\mu\sigma}P_{\rho})\,,\\
[L_{\mu\nu}, L_{\rho\sigma}] &= -i\,(g_{\mu\rho}L_{\nu\sigma} - g_{\mu\sigma}L_{\nu\rho} + g_{\nu\sigma}L_{\mu\rho} - g_{\nu\rho}L_{\mu\sigma})\,.
\end{aligned}$$

In terms of $\mathbf{J}$ and $\mathbf{K}$, the commutation relations read:

$$\begin{aligned}
[P_0\,,\ J_i] &= 0\,,\\
[P_i\,, J_j] &= i\,\varepsilon_{ijk}\,P_k\,,\\
[P_0\,,\ K_i] &= i\,P_i\,,\\
[P_i\,,\ K_j] &= i\,P_0\,\delta_{ij}\,.
\end{aligned}$$

Exercise: Prove all commutation relations that appear on this page.

– Single Particle States

The Poincaré group has two Casimir operators: $P^2 = P^\mu P_\mu$ and $W^2 = W^\mu W_\mu$, where

$$W_\mu \;=\; -\frac{1}{2}\,\varepsilon_{\mu\nu\rho\sigma}\,L^{\nu\rho}\,P^\sigma\,,$$

with $\varepsilon_{0123} = 1$, is the so-called *Pauli–Lubanski* vector.

Classification of $massive$ particle states

A single *massive* particle state $|a\rangle$ can be characterized by its mass and its total spin s, where s is defined in the rest of mass system of the particle:

$$P^2\,|a\rangle \;=\; m^2\,|a\rangle\,,\quad W^2\,|a\rangle \;=\; -m^2\mathbf{J}^2\,|a\rangle \;=\; -m^2 s(s+1)|a\rangle$$

In addition, we use the 3-momentum $\mathbf{P}$ and the *helicity* $H = \mathbf{J}\cdot\mathbf{P}$ operators to classify *massive* particle states:

$$\begin{aligned} P_\mu\,|a;m,s;\mathbf{p},\lambda\rangle &= p_\mu\,|a;m,s;\mathbf{p},\lambda\rangle\,,\\ H\,|a;m,s;\mathbf{p},\lambda\rangle &= \lambda|\mathbf{p}|\,|a;m,s;\mathbf{p},\lambda\rangle\,. \end{aligned}$$

Note that a massive particle state has $(2s+1)$ polarizations or helicities, also called degrees of freedom,
i.e. $\lambda = -s, -s+1, \ldots, s-1, s$.

Examples: for an electron, it is $\lambda = \pm\frac{1}{2}$, and for a massive spin-1 boson (e.g. the Z-boson), we have $\lambda = -1, 0, 1$.

Classification of $massless$ particle states

Massless particle states, for which $P^2|a\rangle = 0$ $(m = 0)$, are characterized only by their 4-momentum p_μ and helicity $\lambda = \mathbf{P}\cdot\mathbf{J}$.

Alternatively, in addition to the operator P_μ, one may use the Pauli–Lubanski operator W_μ:

$$W_\mu|a;p_\mu,\lambda\rangle \;=\; \lambda\,p_\mu\,|a;p_\mu,\lambda\rangle\,.$$

If the theory involves parity, then a massless state has only two degrees of freedom (polarizations): $\pm\lambda$.

Examples of the above are the photon and the neutrinos of the Standard Model.

...

Exercises:

(i) Show that P^2 and W^2 are true Casimir operators, i.e. $[\,P^2,\;P_\mu\,] = [\,P^2,\;L_{\rho\sigma}\,] = 0$, and likewise for W^2;

(ii) In particle's rest frame where $p_\mu = (m, 0, 0, 0)$, show that $W_0 = 0$, $W_i = \frac{1}{2}m\varepsilon_{ijk}\,L^{jk} = mJ_i$ and $W^2 = -m^2\mathbf{J}^2$;

(iii) Show that $[\mathbf{J}\cdot\mathbf{P}, \mathbf{P}] = 0$, $[P_\mu, W_\nu] = 0$, and $W_\mu P^\mu = 0$;

(iv) Calculate the commutation relation $[W_\mu, W_\nu]$.

7. Lagrangians in Field Theory

– Variational Principle and Equation of Motion

Classical Lagrangian Dynamics

The Lagrangian for an n-particle system is

$$L(q_i, \dot{q}_i) \; = \; T \; - \; V \, ,$$

where $q_{1,2,\ldots,n}$ are the the generalized coordinates describing the n particles, and $\dot{q}_{1,2,\ldots,n}$ are the respective time derivatives.

T and V denote the total kinetic and potential energies.

The action S of the n-particle system is given by

$$S[q_i(t)] \; = \; \int_{t_1}^{t_2} dt \, L(q_i, \dot{q}_i) \, .$$

Note that S is a *functional* of $q_i(t)$.

Hamilton's principle

Hamilton's principle states that the actual motion of the system is determined by the stationary behaviour of S under small variations $\delta q_i(t)$ of the ith particle's generalized coordinate $q_i(t)$, with $\delta q_i(t_1) = \delta q_i(t_2) = 0$, i.e.

$$\begin{aligned} \delta S \; &= \; \int_{t_1}^{t_2} dt \left(\delta q_i \frac{\partial L}{\partial q_i} \; + \; \delta \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} \right) \\ &= \; \int_{t_1}^{t_2} dt \, \delta q_i \left(\frac{\partial L}{\partial q_i} \; - \; \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) \; = \; 0 \, . \end{aligned}$$

The Euler–Lagrange equation of motion for the ith particle is

$$\therefore \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \; - \; \frac{\partial L}{\partial q_i} \; = \; 0 \, .$$

$$\cdots$$

Exercise: Show that the Euler–Lagrange equations of motion for a particle system described by a Lagrangian of the form $L(q_i, \dot{q}_i, \ddot{q}_i)$ are

$$\frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{q}_i} \; - \; \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \; + \; \frac{\partial L}{\partial q_i} \; = \; 0 \, .$$

[Hint: Consider only variations with $\delta q_i(t_{1,2}) = \delta \dot{q}_i(t_{1,2}) = 0$.]

Lagrangian Field Theory

In Quantum Field Theory (QFT), a (scalar) particle is described by a field $\phi(x)$, whose Lagrangian has the functional form:

$$L = \int d^3x \; \mathcal{L}(\phi(x)\,, \partial_\mu \phi(x))\,,$$

where $\mathcal{L}$ is the so-called *Lagrangian density*, often termed Lagrangian in QFT.

In QFT, the action S is given by

$$S[\phi(x)] = \int_{-\infty}^{+\infty} d^4x \; \mathcal{L}(\phi(x)\,, \partial_\mu \phi(x))\,,$$

with $\lim\limits_{x \to \pm\infty} \phi(x) = 0$.

By analogy, the Euler–Lagrange equations can be obtained by determining the stationary points of S, under variations $\phi(x) \to \phi(x) + \delta\phi(x)$:

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = 0\,.$$

...

Exercise: Derive the above Euler–Lagrange equation for a scalar particle by extremizing $S[\phi(x)]$, i.e. $\delta S = 0\,.$

– Lagrangians for the Klein-Gordon and Maxwell eqs

Lagrangian for the Klein–Gordon equation

$$\mathcal{L}_{\rm KG} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2\,,$$

where $\phi(x)$ is a real scalar field describing one dynamical degree of freedom.

The Euler–Lagrange equation of motion is the Klein–Gordon equation

$$(\partial_\mu \partial^\mu + m^2)\,\phi(x) = 0\,.$$

...

Lagrangian for the Maxwell equations

$$\mathcal{L}_{\rm ME} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J_\mu A^\mu\,,$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor, and J_μ is the 4-vector current satisfying charge conservation: $\partial_\mu J^\mu = 0$.

A_μ describes a spin-1 particle, e.g. a photon, with 2 physical degrees of freedom.

Exercise: Use the Euler-Lagrange equations for $\mathcal{L}_{\rm ME}$ to show that $\partial_\mu F^{\mu\nu} = J^\nu$, as is expected in relativistic electrodynamics (with $\mu_0 = \varepsilon_0 = c = 1$).

– Lagrangian for the Dirac equation

$$\mathcal{L}_{\rm D} = \bar\psi\,(i\,\gamma^\mu\partial_\mu - m)\,\psi\,,$$

where

$$\psi(x) = \begin{pmatrix} \xi_\beta(x) \\ \bar\eta^{\dot\beta}(x) \end{pmatrix}, \quad \gamma^\mu = \begin{pmatrix} 0 & (\sigma^\mu)_{\alpha\dot\beta} \\ (\bar\sigma^\mu)^{\dot\alpha\beta} & 0 \end{pmatrix}$$

and $\bar\psi(x) \equiv (\eta^\alpha(x),\ \bar\xi_{\dot\alpha}(x))$, with $\sigma^\mu = (\mathbf{1}_2,\ \boldsymbol{\sigma})$ and $\bar\sigma^\mu = (\mathbf{1}_2,\ -\boldsymbol{\sigma})$.

The ξ_α and $\bar\eta^{\dot\alpha}$ are 2-dim complex vectors (also called Weyl spinors) whose components anti-commute: $\xi_1\xi_2 = -\xi_2\xi_1$, $\bar\eta^{\dot 1}\bar\eta^{\dot 2} = -\bar\eta^{\dot 2}\bar\eta^{\dot 1}$, $\xi_1\bar\eta^{\dot 2} = -\bar\eta^{\dot 2}\xi_1$ etc.

The Euler–Lagrange equation of $\mathcal{L}_{\rm D}$ with respect to $\bar\psi$ is the Dirac equation:

$$\frac{\partial\mathcal{L}_{\rm D}}{\partial\bar\psi} = 0 \;\Rightarrow\; (i\,\gamma^\mu\partial_\mu - m)\,\psi = 0\,.$$

The 4-component Dirac spinor $\psi(x)$ that satisfies the Dirac equation describes 4 dynamical degrees of freedom.

Exercises:
(i) Derive the Euler–Lagrange equation with respect to the Dirac field $\psi(x)$;
(ii) Show that up to a total derivative term, $\mathcal{L}_{\rm D}$ is Hermitian, i.e. $\mathcal{L}_{\rm D} = \mathcal{L}_{\rm D}^\dagger + \partial^\mu j_\mu$, with $j_\mu = \bar\psi\, i\gamma_\mu\,\psi$.

Lorentz trans properties of the Weyl and Dirac spinors

The Dirac spinor ψ is the direct sum of two Weyl spinors ξ and $\bar\eta$ with Lorentz trans properties:

$$\xi'_\alpha = M_\alpha^{\ \beta}\,\xi_\beta\,, \qquad \bar\eta'_{\dot\alpha} = M^{\dagger\,\dot\beta}_{\ \ \dot\alpha}\,\bar\eta_{\dot\beta}\,,$$
$$\xi'^\alpha = M^{-1\ \alpha}_{\ \beta}\,\xi^\beta\,, \qquad \bar\eta'^{\dot\alpha} = M^{\dagger-1\,\dot\alpha}_{\ \ \ \ \dot\beta}\,\bar\eta^{\dot\beta}\,.$$

with $M \in {\rm SL}(2,\mathbb{C})$.

Duality relations among 2-spinors:

$$(\xi^\alpha)^\dagger = \bar\xi^{\dot\alpha}\,, \quad (\xi_\alpha)^\dagger = \bar\xi_{\dot\alpha}\,, \quad (\bar\eta_{\dot\alpha})^\dagger = \eta_\alpha\,, \quad (\eta^\alpha)^\dagger = \bar\eta^{\dot\alpha}$$

Lowering and raising spinor indices:

$$\xi_\alpha = \varepsilon_{\alpha\beta}\xi^\beta\,, \quad \xi^\alpha = \varepsilon^{\alpha\beta}\xi_\beta\,, \quad \bar\eta_{\dot\alpha} = \varepsilon_{\dot\alpha\dot\beta}\bar\eta^{\dot\beta}\,, \quad \bar\eta^{\dot\alpha} = \varepsilon^{\dot\alpha\dot\beta}\bar\eta_{\dot\beta}\,,$$

with $\varepsilon^{\alpha\beta} \equiv i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\varepsilon_{\alpha\beta}$ and $\varepsilon^{\dot\alpha\dot\beta} \equiv i\sigma_2 = -\varepsilon_{\dot\alpha\dot\beta}$.
Lorentz-invariant spinor contractions:

$$\xi\eta \equiv \xi^\alpha\eta_\alpha = \xi^\alpha\varepsilon_{\alpha\beta}\eta^\beta = -\eta^\beta\varepsilon_{\alpha\beta}\xi^\alpha = \eta^\beta\varepsilon_{\beta\alpha}\xi^\alpha = \eta^\beta\xi_\beta = \eta\xi$$

Likewise, $\bar\xi\bar\eta \equiv (\eta\xi)^\dagger = \xi^\dagger_\alpha\eta^{\alpha\dagger} = \bar\xi_{\dot\alpha}\bar\eta^{\dot\alpha} = \bar\eta_{\dot\alpha}\bar\xi^{\dot\alpha} = \bar\eta\bar\xi$.

Exercise: Given that $M\sigma_\mu M^\dagger = \Lambda^\nu_{\ \mu}\,\sigma_\nu$ and $M^{\dagger-1}\bar\sigma_\mu M^{-1} = \Lambda^\nu_{\ \mu}\,\bar\sigma_\nu$, show that $\mathcal{L}_D$ is invariant under Lorentz trans.

8. Gauge Groups

– Global and Local Symmetries

Symmetries in Classical Physics and Quantum Mechanics:

Translational invariance in time — Energy conservation

$t \to t + a_0 \quad \Rightarrow \quad \frac{dE}{dt} = 0$

Translational invariance in space — Momentum conservatio

$\mathbf{r} \to \mathbf{r} + \mathbf{a} \quad \Rightarrow \quad \frac{d\mathbf{p}}{dt} = 0$

Rotational invariance — Angular momentum conservation

$\mathbf{r} \to R\,\mathbf{r} \quad \Rightarrow \quad \frac{d\mathbf{J}}{dt} = 0$

Quantum Mechanics — Degeneracy of energy states

$[H, \mathcal{O}] = 0 \quad \Rightarrow \quad \frac{d\mathcal{O}}{dt} = i[H, \mathcal{O}] = 0$

Quantum Field Theory — Noether's Theorem

$\phi(x) \to \phi(x) + \delta\phi(x) \quad \Rightarrow$ **?**

Global and Local Symmetries in QFT

Consider the Lagrangian (density) for a complex scalar:

$$\mathcal{L} \;=\; (\partial^\mu \phi)^* \,(\partial_\mu \phi) \;-\; m^2\, \phi^* \phi \;+\; \lambda(\phi^*\phi)^2 \;.$$

$\mathcal{L}$ is invariant under a U(1) rotation of the field ϕ:

$$\phi(x) \;\to\; \phi'(x) \;=\; e^{i\theta}\, \phi(x) \;,$$

where θ does not depend on $x \equiv x^\mu$.

A transformation in which the fields are rotated about x-independent angles is called a *global transformation*. If the angles of rotation depend on x, the transformation is called a *local* or a *gauge transformation*.

A general infinitesimal global or local trans of fields ϕ_i under the action of a Lie group reads:

$$\phi_i(x) \;\to\; \phi_i'(x) \;=\; \phi_i(x) + \delta\phi_i(x)\,,$$

where $\delta\phi_i(x) \;=\; -i\,\theta^a(x)\;(T^a)_i^{\,j}\,\phi_j(x)\,,$ and T^a are the generators of the Lie Group. Note that the angles or group parameters θ^a are x-independent for a global trans.

If a Lagrangian $\mathcal{L}$ is invariant under a global or local trans, it is said that $\mathcal{L}$ has a *global* or *local (gauge) symmetry*.

Exercise: Show that the above Lagrangian for a complex scalar is *not* invariant under a U(1) gauge trans.

– Gauge Invariance of the QED Lagrangian

Consider first the Lagrangian for a Dirac field ψ:

$$\mathcal{L}_{\rm D} \;=\; \bar{\psi}\,(i\gamma^\mu\partial_\mu \,-\, m)\,\psi\,.$$

$\mathcal{L}_{\rm D}$ is invariant under the U(1) global trans:

$$\psi(x) \;\to\; \psi'(x) \;=\; e^{i\theta}\,\psi(x)\,,$$

but it is *not* invariant under a U(1) gauge trans, when $\theta = \theta(x)$. Instead, we find the residual term

$$\delta\mathcal{L}_{\rm D} \;=\; -\,(\partial_\mu\theta(x))\;\bar{\psi}\gamma^\mu\psi\,.$$

To cancel this term, we introduce a vector field A^μ in the theory, the so-called photon, and add to $\mathcal{L}_D$ the extra term:

$$\mathcal{L}_\psi \;=\; \mathcal{L}_{\rm D} \;-\; e\,A_\mu\,\bar{\psi}\gamma^\mu\psi\,.$$

We demand that A_μ transforms under a local U(1) as

$$A_\mu \;\to\; A'_\mu \;=\; A_\mu \;-\; \frac{1}{e}\,\partial_\mu\theta(x)\,.$$

$\mathcal{L}_\psi$ is invariant under a U(1) gauge trans of ψ and A^μ.

QED Lagrangian with an electron-photon interaction

The complete Lagrangian of Quantum Electrodynamics (QED) that includes the interaction of the photon with the electron is

$$\mathcal{L}_{\rm QED} \;=\; -\frac{1}{4}\,F_{\mu\nu}\,F^{\mu\nu} \;+\; \bar{\psi}\,(i\,\partial\!\!\!/ \;-\; m \;-\; e\,A\!\!\!/)\,\psi\,,$$

where we used the convention: $a\!\!\!/ \equiv \gamma_\mu a^\mu$.

Exercises:

(i) Show that $\mathcal{L}_{\rm QED}$ is gauge invariant under a U(1) trans.

(ii) Derive the equation of motions with respect to photon and electron fields.

(iii) How should the Lagrangian describing a complex scalar field $\phi(x)$,

$$\mathcal{L} \;=\; (\partial^\mu\phi)^*\,(\partial_\mu\phi) \;-\; m^2\,\phi^*\phi\,,$$

be extended so as to become gauge symmetric under a U(1) local trans?

– Noether's Theorem

Noether's Theorem. If a Lagrangian $\mathcal{L}$ is symmetric under a global transformation of the fields, then there is a conserved current $J^\mu(x)$ and a conserved charge $Q = \int d^3x\, J^0(x)$, associated with this symmetry, such that

$$\partial_\mu J^\mu = 0 \quad \text{and} \quad \frac{dQ}{dt} = 0\,.$$

Proof:

Consider a Lagrangian $\mathcal{L}(\phi_i, \partial_\mu\phi_i)$ to be invariant under the infinitesimal global trans:

$$\delta\phi_i = i\,\theta^a (T^a)_i^{\;j}\,\phi_j\,,$$

where T^a are the generators of some group G.

Hence, the change of $\mathcal{L}$ is vanishing, i.e.

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi_i}\,\delta\phi_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\,\partial_\mu(\delta\phi_i) = 0\,.$$

This last equation can be rewritten as

$$\delta\mathcal{L} = \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\,\delta\phi_i\right] + \left[\frac{\partial\mathcal{L}}{\partial\phi_i} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\right]\delta\phi_i = 0\,.$$

With the aid of the equations of motions for ϕ_i, the last equation implies that

$$\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\,\delta\phi_i\right] = \left[\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)} - \frac{\partial\mathcal{L}}{\partial\phi_i}\right]\delta\phi_i = 0\,.$$

The conserved current (or currents) is

$$J^{a,\,\mu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\,\frac{\partial\delta\phi_i}{\partial\theta^a} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\,i\,(T^a)_i^{\;j}\,\phi_j\,.$$

The corresponding conserved charges are

$$Q^a(t) = \int d^3x\, J^{a,\,0}(x)\,.$$

Indeed, it is easy to check that

$$\begin{aligned}\frac{dQ^a}{dt} &= \int d^3x\;\partial_0\, J^{a,\,0}(x) = -\int d^3x\;\nabla\cdot\mathbf{J}^a(x)\\ &= -\int d\mathbf{s}\cdot\mathbf{J}^a \;\to\; 0\,,\end{aligned}$$

because surface terms vanish at infinity.

Exercises: Find the conserved currents and charges for
(i) QED;
(ii) the gauge-invariant Lagrangian with a complex scalar ϕ.

– Yang–Mills Theory

The Lagrangian of a Yang–Mills (non-Abelian) SU(N) theory is

$$\mathcal{L}_{\rm YM} \;=\; -\frac{1}{4}\, F^a_{\mu\nu}\, F^{a,\mu\nu}\,,$$

where

$$F^a_{\mu\nu} \;=\; \partial_\mu A^a_\nu \,-\, \partial_\nu A^a_\mu \,-\, g\, f^{abc}\, A^b_\mu\, A^c_\nu\,,$$

and f^{abc} are the structure constants of the SU(N) Lie algebra.

It can be shown that $\mathcal{L}_{\rm YM}$ is invariant under the infinitesimal SU(N) local trans:

$$\delta A^a_\mu \;=\; -\frac{1}{g}\,\partial_\mu\theta^a \,-\, f^{abc}\,\theta^b\, A^c_\mu\,.$$

Examples of SU(N) theories are the SU$(2)_L$ group of the SM and Quantum Chromodynamics (QCD) based on the SU$(3)_c$ group.

The gauge (vector) fields of the SU$(2)_L$ are the W^0 and $W^\pm$ bosons responsible for the weak force.

The gauge vector bosons of the SU$(3)_c$ group are the gluons mediating the strong force between quarks.

Gauge bosons of Yang–Mills theories self-interact!

Exercise: Show that $\mathcal{L}_{\rm YM}$ is invariant under SU(N) gauge trans.

Interaction between quarks q_i and gluons A^a_μ in SU(3)$_c$

If $q_i = (q_{\rm red},\; q_{\rm green},\; q_{\rm blue})$ are the 3 colours of the quark, their interaction with the 8 gluons A^a_μ is described by the Lagrangian:

$$\mathcal{L}_{\rm int} \;=\; \bar{q}^i\,[\, i\,\partial\!\!\!/\,\delta^j_i \,-\, m\,\delta^j_i \,-\, g\, A\!\!\!/^a (T^a)^{\;j}_i\,]\, q_j\,.$$

Exercise: Show that $\mathcal{L}_{\rm int}$ is invariant under the SU(3) gauge transformation:

$$\delta A^a_\mu \;=\; -\frac{1}{g}\,\partial_\mu\theta^a \,-\, f^{abc}\,\theta^b\, A^c_\mu\,, \qquad \delta q_i \;=\; i\theta^a\,(T^a)^{\;j}_i\, q_j\,,$$

where $T^a = \frac{1}{2}\,\lambda^a$ are the generators of SU(3) and λ^a are the Gell-Mann matrices:

$$\lambda^{1,2,3} \;=\; \begin{pmatrix} \sigma^{1,2,3} & 0 \\ 0 & 0 \end{pmatrix}, \qquad \lambda^4 \;=\; \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

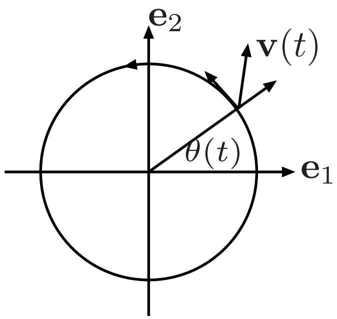
$$\lambda^5 \;=\; \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \qquad \lambda^6 \;=\; \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda^7 \;=\; \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad \lambda^8 \;=\; \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & -\frac{2}{\sqrt{3}} \end{pmatrix}.$$

9. The Geometry of Gauge Transformations

– Parallel Transport and Covariant Derivative

Simple Example:



Time-dependent vector written in terms of t-dependent unit vectors:

$$\mathbf{v}(t) \;=\; v_i(t)\,\mathbf{e}_i(t) \qquad \text{(with } i = 1,2\text{)}.$$

The *true* time derivative of $\mathbf{v}(t)$ is

$$\frac{d}{dt}\mathbf{v}(t) \;=\; \lim_{\delta t \to 0} \frac{\mathbf{v}(t+\delta t) \;-\; \mathbf{v}(t)}{\delta t}\;.$$

To calculate this, we need to refer all unit vectors to $t + \delta t$:

$$\mathbf{e}_i(t) \;=\; \mathbf{e}_i(t+\delta t) \;-\; \delta t\,\partial_t \mathbf{e}_i(t)\,.$$

Then, we have

$$\begin{aligned} \frac{d}{dt}\mathbf{v}(t) \;&=\; \frac{1}{\delta t}\,\{\,v_i(t+\delta t) \;-\; (v_i(t) \;-\; \delta t\, v_j(t)\,[\mathbf{e}_i\cdot\partial_t\mathbf{e}_j])\}\\ &\quad \times \mathbf{e}_i(t+\delta t)\\ &=\; [\,\partial_t v_i(t) \;+\; (\mathbf{e}_i\cdot\partial_t\mathbf{e}_j)\,v_j(t)\,]\;\mathbf{e}_i(t)\,. \end{aligned}$$

We can now define the *covariant derivative* to act *only* on the components of $\mathbf{v}(t)$ as:

$$\begin{aligned} D_t v_i(t) \;&=\; \partial_t v_i(t) \;+\; (\mathbf{e}_i\cdot\partial_t\mathbf{e}_j)v_j(t)\,,\\ &=\; \partial_t v_i(t) \;+\; \dot\theta\,\varepsilon_{3ij}\,v_j(t)\,, \end{aligned}$$

with the obvious property $\frac{d}{dt}\mathbf{v}(t) = \mathbf{e}_i(t)D_t v_i(t)$. The second term is induced by the change of the coordinate axes, namely after performing a *parallel transport* of our coordinate system $\mathbf{e}_{1,2}(t)$ from t to $t+\delta t$.

Proper comparison of two vectors $v_i(t+\delta t)$ and $v_i(t)$ can only be made in the same coordinate system by means of *parallel transport*. Differentiation is properly defined through the covariant derivative.

Exercise: Show that the covariant derivative satisfies the relation

$$D_t v_i(t) \;=\; \partial_t v_i(t) \;+\; (\boldsymbol{\omega}\times\mathbf{v}(t))_i\,,$$

with $\omega = \dot\theta(t)$, which is known from Classical Mechanics between rotating and fixed frames in 3 dimensions.

Differentiation in curved space

The notion of the covariant derivative generalizes to curved space as well. By analogy, the infinitesimal difference between the 4-vectors $V^\mu(x'^\mu)$ and $V(x^\mu)$ is given by

$$DV^\mu \;=\; dV^\mu \;+\; \delta V^\mu \,,$$

where dV^μ is the difference of the 2 vectors in the same coordinate system and δV^μ is due to parallel transport of the vector from x^μ to $x'^\mu = x^\mu + \delta x^\mu$.

In the framework of General Relativity, we have

$$DV^\mu \;=\; (\partial_\lambda V^\mu \;+\; \Gamma^\mu_{\nu\lambda} V^\nu)\; dx^\lambda \,,$$

where $\Gamma^\mu_{\nu\lambda}$ is the so-called *affine connection* or the *Christoffel symbol*.

Covariant derivative in the Gauge-Group Space

Consider the difference of a fermionic *isovector* field ψ at $x^\mu + \delta x^\mu$ and x^μ in an SU(N) gauge theory:

$$D\psi \;=\; d\psi \;+\; \delta\psi \,,$$

where

$$\delta\psi \;=\; ig\, T^a A^a_\mu\, dx^\mu\, \psi$$

and the field A^a_μ takes care of the change of the SU(N) axes from point to point in Minkowski space.

The covariant derivative of $\psi(x^\mu)$ is

$$D_\mu\psi \;=\; (\partial_\mu \;+\; ig\, T^a A^a_\mu)\, \psi \,,$$

which is obtained from pure geometric considerations.

In analogy to General Relativity, the gauge field $A^a_\mu T^a$ is sometimes called the *connection*.

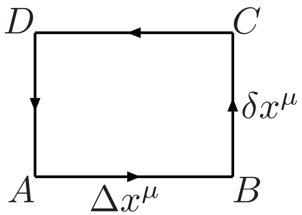
Exercise: Show that under a local SU(N) rotation of the *isovector* ψ field: $\psi \to \psi' = U\psi$ (with $U \in$ SU(N)), its covariant derivative transforms as

$$D_\mu\psi \;\to\; D'_\mu\psi' \;=\; U\, D_\mu\psi \,,$$

with

$$A'_\mu \;=\; U\, A_\mu\, U^\dagger \;+\; \frac{i}{g}\, (\partial_\mu U)\, U^\dagger \,.$$

A round trip in the SU(N) Gauge-Group Space



Keeping terms up to second order in δx and Δx, we have

$$\begin{aligned}
\psi_B &= (1 + \Delta x^\mu D_\mu + \frac{1}{2}\Delta x^\mu \Delta x^\nu D_\mu D_\nu)\,\psi_{A,0}\,,\\
\psi_C &= (1 + \delta x^\mu D_\mu + \frac{1}{2}\delta x^\mu \delta x^\nu D_\mu D_\nu)\,\psi_B\,,\\
\psi_D &= (1 - \Delta x^\mu D_\mu + \frac{1}{2}\Delta x^\mu \Delta x^\nu D_\mu D_\nu)\,\psi_C\,,\\
\psi_{A,1} &= (1 - \delta x^\mu D_\mu + \frac{1}{2}\delta x^\mu \delta x^\nu D_\mu D_\nu)\,\psi_D\,.
\end{aligned}$$

Hence,

$$\psi_{A,1} = (1 + \delta x^\mu \Delta x^\nu [D_\mu\,,\,D_\nu])\,\psi_{A,0}\,,$$

and $\psi_{A,1} \neq \psi_{A,0}$.

Exercise: Show that

$$\frac{i}{g}[D_\mu\,,\,D_\nu] = F^a_{\mu\nu}\,T^a$$

is the SU(N) Field-strength tensor.

Parallels between Gauge Theory and General Relativity

In General Relativity, a corresponding round trip of a vector V^μ in a curved space gives rise to

$$\Delta V^\mu = \frac{1}{2} R^\mu_{\rho\sigma\lambda}\, V^\rho \Delta S^{\sigma\lambda}\,,$$

where $\Delta S^{\sigma\lambda}$ represents the area enclosed by the path and $R^\mu_{\rho\sigma\lambda}$ is the Riemann–Christoffel *curvature tensor*:

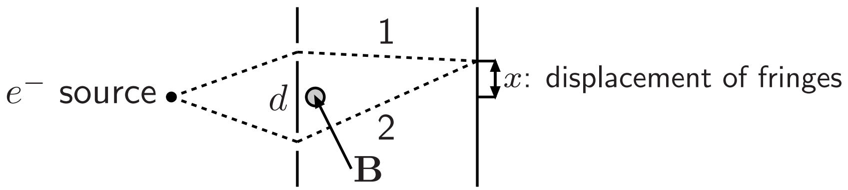
$$R^\mu_{\rho\sigma\lambda} = \partial_\lambda \Gamma^\mu_{\rho\sigma} - \partial_\sigma \Gamma^\mu_{\rho\lambda} + \Gamma^\kappa_{\rho\sigma}\Gamma^\mu_{\kappa\lambda} - \Gamma^\kappa_{\rho\lambda}\Gamma^\mu_{\kappa\sigma}\,.$$

Analogies:

Gauge Theory	General Relativity
Gauge trans.	Co-ordinate trans.
Gauge field $A^a_\mu T^a$	Affine connection, $\Gamma^\kappa_{\lambda\nu}$
Field strength $F^{\mu\nu}$	Curvature tensor $R^\mu_{\rho\sigma\lambda}$
Bianchi identity:	Bianchi identity:
$\sum_{\substack{\rho,\mu,\nu\\ \text{cyclic}}} D_\rho F_{\mu\nu} = 0$	$\sum_{\substack{\rho,\mu,\nu\\ \text{cyclic}}} D_\rho R^\kappa_{\lambda\mu\nu} = 0$

– Topology of the Vacuum: the Bohm–Aharanov Effect

The Bohm–Aharanov Effect:



Vector potential $\mathbf{A}$ and $\mathbf{B}$ field (with $\mathbf{B} = \nabla \times \mathbf{A}$) in cylindrical polars:

$$\begin{aligned} \textit{Inside:} \quad & A_r = A_z = 0, \quad A_\phi = \frac{Br}{2}, \\ & B_r = B_\phi = 0, \quad B_z = B, \\ \textit{Outside:} \quad & A_r = A_z = 0, \quad A_\phi = \frac{BR^2}{2r}, \\ & \mathbf{B} = 0, \end{aligned}$$

where R is the radius of the solenoid.

Although the electrons move in regions with $\mathbf{E} = \mathbf{B} = \mathbf{0}$, the $\mathbf{B}$ field of the solenoid induces a phase difference $\delta\phi_{12}$ of the electrons on the screen causing a displacement of the fringes:

$$\delta\phi_{12} \; = \; \phi_1 - \phi_2 \; = \; \frac{e}{\hbar} \oint_{2-1} \mathbf{A} \cdot d\mathbf{r} \; = \; \frac{e}{\hbar} \int \mathbf{B} \cdot d\mathbf{s} \, .$$

In regions with $\mathbf{E} = \mathbf{B} = \mathbf{0}$, it is $\mathbf{A} \neq 0$, so the vacuum has a *topological* structure! It is not *simply* connected due to the presence of the solenoid.

Basic Concepts in Topology

Let $a(s)$ and $b(s)$ be two paths in a topological space Y both starting from the point P ($a(0) = b(0) = P$) and ending at a possibly different point Q ($a(1) = b(1) = Q$). If there exists a *continuous* function $L(t, s)$ such that $L(0, s) = a(s)$ and $L(1, s) = b(s)$, then the paths a and b are called *homotopic* which is denoted by $a \sim b$.

If $P \equiv Q$, the path is said to be *closed*.

The inverse of a path a is written as a^{-1} and is defined by $a^{-1}(s) = a(1 - s)$. It corresponds to the same path traversed in the opposite direction.

The *product path* $c = ab$ is defined by

$$\begin{aligned} c(s) &= a(2s)\,, && \text{for } 0 \leq s \leq \tfrac{1}{2}\,, \\ c(s) &= b(2s - 1)\,, && \text{for } \tfrac{1}{2} \leq s \leq 1\,. \end{aligned}$$

If $a \sim b$, then ab^{-1} is homotopic to the *null path*: $ab^{-1} \sim 1$.

Exercises:

(i) Consider the mappings $S^1 \to U(1)$: $f_n(\theta) = e^{i(n\theta + a)}$ (with $a \in \mathbb{R}$ and $n \in \mathbb{Z}$), and show that they all are homotopic to those with $a = 0$.

(ii) Given that $f_n(\theta) \not\sim f_m(\theta)$ for $n \neq m$, explain then why $L(t, \theta) = e^{i[n\theta(1-t) + m\theta t]}$ is not an allowed homotopy function relating f_n to f_m.

Homotopy Classes, Groups and the Winding Number

All paths related to maps $X \to Y$ of two topological spaces X, Y can be divided into *homotopy classes*.

Homotopy Class. All paths that are homotopic to a given path $a(s)$ define a set, called the *homotopy class* and denoted by $[a]$. For example, $[f_n]$ are distinct homotopy classes for different n.

Winding Number. Each homotopy class may be characterized by an integer, the *winding number* n (also called the *Pontryargin index*). For the case $f(\theta):\ S^1 \to U(1)$, the winding number is determined by

$$n \;=\; \frac{1}{2\pi i}\int_0^{2\pi} d\theta\,\left(\frac{d\ln f(\theta)}{d\theta}\right)\,.$$

Homotopy Group. The set of all homotopy classes related to maps $X \to Y$ forms a group, under the multiplication law

$$[a]\,[b] \;=\; [ab]\,,$$

the so-called *homotopy group* $\pi_X(Y)$.

Exercises:
(i) Prove that the homotopy group satisfies the axioms of a group.

(ii) Show that for $S^1 \to U(1)$, $\pi_1[U(1)] \cong \mathbb{Z}$.

The Bohm–Aharanov Effect Revisited

In regions with $\mathbf{E} = \mathbf{B} = \mathbf{0}$, A_μ is a pure gauge: $A_\mu = \partial_\mu \chi$ (*Why?*).

The configuration space X of the Bohm–Aharanov effect is the plane $\mathbb{R}^2$ with a hole in it, due to the solenoid. This is topologically equivalent ($\equiv$ *homeomorphic*) to $\mathbb{R} \times S^1$. The space X can be conveniently described by polar coords (r, ϕ), with $r \neq 0$.

It can be shown that $\chi(r,\phi) = \mathrm{const.} \times \phi$, which is a function in the group space of U(1), i.e. $Y = U(1)$.

Since functions mapping S^1 onto $\mathbb{R}$ are all deformable to a constant, the non-trivial part of χ is given by the map:

$$S^1 \;\to\; U(1)\,.$$

Because $\pi_1[U(1)] = \mathbb{Z}$, the electron paths cannot be deformed to a null path with a constant χ, implying $A_\mu = 0$ everywhere and the absence of the Bohm–Aharanov effect.

Since $\pi_1[SU(2)] \;=\; 1$, there is *no* Bohm–Aharanov effect from an SU(2) 'solenoid'!

Exercises:
(i) Show that $\chi(r,\phi) = \frac{1}{2} BR^2\,\phi$ is a possible solution for $\mathbf{E} = \mathbf{B} = \mathbf{0}$, where B is the magnetic field and R the radius of the solenoid.

(ii) Verify that $\delta\phi_{12} = \frac{e}{\hbar}\,[\,\chi(2\pi) - \chi(0)\,]$.

10. Supersymmetry (SUSY)

– Graded Lie Algebra

Definition. A $\mathbb{Z}_2$-graded Lie algebra is defined on a vector space L which is the direct sum of two subspaces L_0 and L_1: $L = L_0 \oplus L_1$. The generators that span the space L are endowed with a multiplications law:

$$\circ : \; L \times L \to L \, .$$

$\forall \, T^{(0)} \in L_0, \; T^{(1)} \in L_1$, the generators satisfy the following properties:

(i) $T_1^{(0)} \circ T_2^{(0)} \; = \; -(-1)^{g_0^2} T_2^{(0)} \circ T_1^{(0)} \; = \; [T_1^{(0)}, \, T_2^{(0)}] \in L_0 \, ,$
(ii) $T^{(0)} \circ T^{(1)} = -(-1)^{g_0 g_1} T^{(1)} \circ T^{(0)} = \{T^{(0)}, \, T^{(1)}\} \in L_1 \, ,$
(iii) $T_1^{(1)} \circ T_2^{(1)} \; = \; -(-1)^{g_1^2} T_2^{(1)} \circ T_1^{(1)} \; = \; [T_1^{(1)}, \, T_2^{(1)}] \in L_0 \, ,$

where $g_0 = g(L_0) = 0$ and $g_1 = g(L_1) = 1$ are the degrees of the graduation of the $\mathbb{Z}_2$-graded Lie algebra.

In addition, all generators of L satisfy the $\mathbb{Z}_2$-graded Jacobi identity:

$$(-1)^{g_i g_k} T^{(i)} \circ (T^{(j)} \circ T^{(k)}) \; + \; (-1)^{g_k g_j} T^{(k)} \circ (T^{(i)} \circ T^{(j)}) \\ + \; (-1)^{g_j g_i} T^{(j)} \circ (T^{(k)} \circ T^{(i)}) \; = \; 0 \, ,$$

where $i, j, k = 0, 1$.

$\mathbb{Z}_N$**-graded Lie algebra**. The generalization of a $\mathbb{Z}_2$-graded Lie algebra L to $\mathbb{Z}_N$ can be defined analogously. Let L be the direct sum of N subalgebras L_i:

$$L \; = \; \oplus_{i=0}^{N-1} \, L_i \, .$$

Then, the multiplication law $\circ$ among the generators of L can be defined by

$$T^{(i)} \circ T^{(j)} \; = \; -(-1)^{g_i g_j} \, T^{(j)} \circ T^{(i)} \; \in \; L_{(i+j) \bmod N} \, .$$

The $\mathbb{Z}_N$-graded Jacobi identity is defined analogously with that of $\mathbb{Z}_2$, where $g_{i,j} = 0, 1, \ldots, N-1$ is the degree of graduation of $L_{i,j}$.

$\cdots$

Exercise:*** Find the (anti)-commutation relations and the structure constants of the $\mathbb{Z}_2$-graded Lie algebra of SU(2).

– Generators of the Super-Poincaré Group

The generators super-Poincaré algebra are $P_\mu\,,\ L_{\mu\nu}\ \in\ L_0$ and the spinors $Q_\alpha\,,\ \bar{Q}_{\dot\alpha}\ \in\ L_1$. They satisfy the following relations:

(i) $[P_\mu\,,\ P_\nu]\ =\ 0\,,$

(ii) $[P_\mu, L_{\rho\sigma}]\ =\ i\,(g_{\mu\rho}P_\sigma - g_{\mu\sigma}P_\rho)\,,$

(iii) $[L_{\mu\nu}, L_{\rho\sigma}]\ =\ -i\,(g_{\mu\rho}L_{\nu\sigma} - g_{\mu\sigma}L_{\nu\rho} + g_{\nu\sigma}L_{\mu\rho} - g_{\nu\rho}L_{\mu\sigma})\,.$

(iv) $\{Q_\alpha,\ Q_\beta\}\ =\ \{\bar{Q}_{\dot\alpha},\ \bar{Q}_{\dot\beta}\}\ =\ 0\,,$

(v) $\{Q_\alpha,\ \bar{Q}_{\dot\beta}\}\ =\ 2(\sigma^\mu)_{\alpha\dot\beta}P_\mu\,,$

(vi) $[Q_\alpha,\ P_\mu]\ =\ 0\,,$

(vii) $[L_{\mu\nu}\,,\ Q_\alpha]\ =\ -i(\sigma_{\mu\nu})_\alpha^{\ \beta}\,Q_\beta\,,$

(viii) $[L_{\mu\nu}\,,\ \bar{Q}_{\dot\alpha}]\ =\ -i(\bar\sigma_{\mu\nu})_{\dot\alpha}^{\ \dot\beta}\,\bar{Q}_{\dot\beta}\,,$

where $(\sigma^{\mu\nu})_\alpha^{\ \beta} = \frac{1}{4}\,[\,(\sigma^\mu)_{\alpha\dot\alpha}(\bar\sigma^\nu)^{\dot\alpha\beta} - (\sigma^\nu)_{\alpha\dot\alpha}(\bar\sigma^\mu)^{\dot\alpha\beta}\,]$ and $(\bar\sigma^{\mu\nu})^{\dot\alpha}_{\ \dot\beta} = \frac{1}{4}\,[\,(\bar\sigma^\mu)^{\dot\alpha\beta}(\sigma^\nu)_{\beta\dot\beta} - (\bar\sigma^\nu)^{\dot\alpha\beta}(\sigma^\mu)_{\beta\dot\beta}\,]$.

...

Exercise:⋆ Prove the $\mathbb{Z}_2$-graded Jacobi identity:

$$[L_{\mu\nu}\,,\ \{Q_\alpha\,,\ \bar{Q}_{\dot\beta}\}] + \{Q_\alpha\,,\ [Q_{\dot\beta}\,,\ L_{\mu\nu}]\} + \{\bar{Q}_{\dot\beta}\,,\ [Q_\alpha\,,\ L_{\mu\nu}]\} = 0\,.$$

Consequences of the Super-Poincaré Symmetry

- Equal number of fermions and bosons.
- Scalar supermultiplet $\widehat{\Phi} \supset (\phi\,,\ \xi\,,\ F)$, where ϕ is a complex scalar (2), ξ is a 2-component complex spinor (4), and F is an auxiliary complex scalar (2).
- Vector supermultiplet $\widehat{V}^a \supset (A^a_\mu\,,\ \lambda^a,\ D^a)$, where A^a_μ are massless non-Abelian gauge fields (3), λ^a are the 2-component gauginos (4), and D^a are the auxiliary real fields (1).

The simplest model that realizes SUperSYmmetry (SUSY) is the Wess–Zumino model. Counting on-shell degrees of freedom (dof), the Wess-Zumino model contains one complex scalar ϕ (2 dofs) and one Weyl spinor ξ (2 dofs):

bosonic dofs $=$ fermionic dofs

– The Wess–Zumino Model

Non-interacting WZ model

$$\mathcal{L}_{\rm kin} = \mathcal{L}_{\rm scalar} + \mathcal{L}_{\rm fermion}$$
$$= (\partial^\mu \phi^\dagger)(\partial_\mu \phi) + \bar{\xi}\, i\bar{\sigma}^\mu(\partial_\mu \xi)\,; \quad \phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$$

Consider $\phi \to \phi + \delta\phi$ and $\phi^\dagger \to \phi^\dagger + \delta\phi^\dagger$, with

$$\delta\phi = \theta\xi \quad \text{and} \quad \delta\phi^\dagger = (\theta\xi)^\dagger = \bar{\xi}\bar{\theta} = \bar{\theta}\bar{\xi}\,,$$

and θ infinitesimal anticommuting 2-spinor constant.

$$\Rightarrow \mathcal{L}_{\rm scalar} \to \mathcal{L}_{\rm scalar} + \delta\mathcal{L}_{\rm scalar}\,,$$
$$\delta\mathcal{L}_{\rm scalar} = \theta(\partial^\mu\phi^\dagger)(\partial_\mu\xi) + \bar{\theta}(\partial^\mu\bar{\xi})(\partial_\mu\phi)$$

Try $\xi_\alpha \to \xi_\alpha + \delta\xi_\alpha$ and $\bar{\xi}_{\dot\alpha} \to \bar{\xi}_{\dot\alpha} + \delta\bar{\xi}_{\dot\alpha}$, with

$$\delta\xi_\alpha = -i(\sigma^\mu\bar{\theta})_\alpha\partial_\mu\phi \quad \text{and} \quad \delta\bar{\xi}_{\dot\alpha} = i(\theta\sigma^\mu)_{\dot\alpha}\partial_\mu\phi^\dagger$$

$$\Rightarrow \mathcal{L}_{\rm fermion} \to \mathcal{L}_{\rm fermion} + \delta\mathcal{L}_{\rm fermion}\,,$$
$$\begin{aligned}\delta\mathcal{L}_{\rm fermion} &= -\,\theta\sigma^\nu\bar{\sigma}^\mu(\partial_\mu\xi)(\partial_\nu\phi^\dagger) + \bar{\xi}\bar{\sigma}^\mu\sigma^\nu\bar{\theta}(\partial_\mu\partial_\nu\phi)\\ &= \theta\sigma^\nu\bar{\sigma}^\mu\xi(\partial_\mu\partial_\nu\phi^\dagger) + \bar{\xi}\bar{\sigma}^\mu\sigma^\nu\bar{\theta}(\partial_\mu\partial_\nu\phi)\\ &\quad -\partial_\mu\left[\theta\sigma^\nu\bar{\sigma}^\mu\xi(\partial_\nu\phi^\dagger)\right]\end{aligned}$$

Exercise: Show that

$$\{\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu\}_\alpha^{\ \beta} = 2g^{\mu\nu}\delta_\alpha^{\ \beta}\,, \quad \{\bar{\sigma}^\mu\sigma^\nu + \bar{\sigma}^\nu\sigma^\mu\}^{\dot\alpha}_{\ \dot\beta} = 2g^{\mu\nu}\delta^{\dot\alpha}_{\ \dot\beta}$$

Noticing that $\partial_\mu\partial_\nu = \partial_\nu\partial_\mu$ and using the results of the above exercise, we get

$$\begin{aligned}\delta\mathcal{L}_{\rm fermion} &= \theta\xi(\partial_\mu\partial^\mu\phi^\dagger) + \bar{\xi}\bar{\theta}(\partial_\mu\partial^\mu\phi)\\ &= -\theta(\partial_\mu\xi)(\partial^\mu\phi^\dagger) - \bar{\theta}(\partial_\mu\bar{\xi})(\partial^\mu\phi)\\ &\quad +\partial_\mu\left[\theta\xi(\partial^\mu\phi^\dagger) + \bar{\xi}\bar{\theta}(\partial^\mu\phi)\right]\end{aligned}$$

$$\Rightarrow \delta\mathcal{L} = \delta\mathcal{L}_{\rm scalar} + \delta\mathcal{L}_{\rm fermion} = 0\ !$$

But, we are not finished yet ! The difference of two successive SUSY transfs. must be a symmetry of the Lagrangian as well, i.e. SUSY algebra should close.

$$\begin{aligned}(\delta_{\theta_2}\delta_{\theta_1} - \delta_{\theta_1}\delta_{\theta_2})\phi &= -i(\theta_1\sigma^\mu\bar{\theta}_2 - \theta_2\sigma^\mu\bar{\theta}_1)\,\partial_\mu\phi\\ &\equiv i\epsilon^\mu P_\mu\phi \quad (\text{with } \epsilon^{\mu*} = \epsilon^\mu)\end{aligned}$$

$$\begin{aligned}(\delta_{\theta_2}\delta_{\theta_1} - \delta_{\theta_1}\delta_{\theta_2})\xi_\alpha &= -i(\sigma^\mu\bar{\theta}_1)_\alpha\theta_2\partial_\mu\xi + i(\sigma^\mu\bar{\theta}_2)_\alpha\theta_1\partial_\mu\xi\\ &\overset{\rm Fierz}{=} -i(\theta_1\sigma^\mu\bar{\theta}_2 - \theta_2\sigma^\mu\bar{\theta}_1)\,\partial_\mu\xi_\alpha\\ &\quad + \theta_{1\alpha}\bar{\theta}_2 i\bar{\sigma}^\mu\partial_\mu\xi - \theta_{2\alpha}\bar{\theta}_1 i\bar{\sigma}^\mu\partial_\mu\xi\end{aligned}$$

Only for on-shell fermions, $i\bar{\sigma}^\mu\partial_\mu\xi = 0$, the SUSY algebra closes.

To close SUSY algebra off-shell, we need an *auxiliary* complex scalar F (without kinetic term) and add

$$\mathcal{L}_F \;=\; F^\dagger F$$

to $\mathcal{L}_{\rm scalar} + \mathcal{L}_{\rm fermion}$, with

$$\begin{array}{ll} \delta F = -i\bar\theta\bar\sigma^\mu(\partial_\mu\xi)\,, & \delta F^\dagger = i(\partial_\mu\bar\xi)\bar\sigma^\mu\theta \\ \delta\xi_\alpha = -i(\sigma^\mu\bar\theta)_\alpha\partial_\mu\phi + \theta_\alpha F\,, & \delta\bar\xi_{\dot\alpha} = i(\theta\sigma^\mu)_{\dot\alpha}\partial_\mu\phi^\dagger + \bar\theta_{\dot\alpha}F^\dagger \end{array}$$

Exercise: Prove **(i)** that the Lagrangian

$$\mathcal{L}_{\rm kin} = (\partial^\mu\phi^\dagger)(\partial_\mu\phi) \;+\; \bar\xi\, i\bar\sigma^\mu(\partial_\mu\xi) \;+\; F^\dagger F$$

is invariant under the off-shell SUSY transfs:

$$\begin{array}{ll} \delta\phi = \theta\xi\,, & \delta\phi^\dagger = \bar\theta\bar\xi \\ \delta\xi_\alpha = -i(\sigma^\mu\bar\theta)_\alpha\partial_\mu\phi + \theta_\alpha F\,, & \delta\bar\xi_{\dot\alpha} = i(\theta\sigma^\mu)_{\dot\alpha}\partial_\mu\phi^\dagger + \bar\theta_{\dot\alpha}F^\dagger \\ \delta F = -i\bar\theta\bar\sigma^\mu(\partial_\mu\xi)\,, & \delta F^\dagger = i(\partial_\mu\bar\xi)\bar\sigma^\mu\theta \end{array}$$

and **(ii)** that the SUSY algebra closes off-shell:

$$(\delta_{\theta_2}\delta_{\theta_1} - \delta_{\theta_1}\delta_{\theta_2})X \;=\; -i(\theta_1\sigma^\mu\bar\theta_2 - \theta_2\sigma^\mu\bar\theta_1)\,\partial_\mu X\,,$$

with $X = \phi,\ \phi^\dagger,\ \xi,\ \bar\xi,\ F,\ F^\dagger$.

The interacting WZ model

$$\begin{aligned} \mathcal{L}_{\rm WZ} &= \mathcal{L}_{\rm kin} \;+\; \mathcal{L}_{\rm int} \\ &= (\partial^\mu\phi^\dagger)(\partial_\mu\phi) \;+\; \bar\xi\, i\bar\sigma^\mu(\partial_\mu\xi) \;+\; F^\dagger F \\ &\quad -\frac{1}{2}W_{\phi\phi}\,\xi\xi \;+\; W_\phi\,F \;-\; \frac{1}{2}W^\dagger_{\phi\phi}\bar\xi\bar\xi \;+\; W^\dagger_\phi F^\dagger \end{aligned}$$

where

$$W(\phi) \;=\; \frac{m}{2}\,\phi\phi \;+\; \frac{h}{6}\,\phi\phi\phi$$

is the so-called superpotential, and

$$\begin{aligned} W_\phi &= \frac{\delta W}{\delta\phi} \;=\; m\phi \;+\; \frac{h}{2}\phi^2 \\ W_{\phi\phi} &= \frac{\delta^2 W}{\delta\phi\,\delta\phi} \;=\; m \;+\; h\phi \end{aligned}$$

Exercise: Show that up to total derivatives,

$$\begin{aligned} \mathcal{L}_{\rm int} &= -\frac{1}{2}W_{\phi\phi}\,\xi\xi \;+\; W_\phi\,F \;-\; \frac{1}{2}W^\dagger_{\phi\phi}\bar\xi\bar\xi \;+\; W^\dagger_\phi F^\dagger \\ &= -\frac{1}{2}\,(m + h\phi)\xi\xi \;-\; \frac{1}{2}\,(m + h\phi^\dagger)\bar\xi\bar\xi \\ &\quad + (m\phi + \frac{h}{2}\phi^2)F \;+\; (m\phi^\dagger + \frac{h}{2}\phi^{\dagger 2})F^\dagger \end{aligned}$$

remains invariant under off-shell SUSY transformations.

– Feynman rules

Equation of motions for the auxiliary fields F and $F^\dagger$:

$$F = -W^\dagger_\phi\,, \qquad F^\dagger = -W_\phi\,,$$

Substituting the above into $\mathcal{L}_{\rm WZ}$, we get

$$\begin{aligned}\mathcal{L}_{\rm WZ} &= (\partial^\mu\phi^\dagger)(\partial_\mu\phi) + \bar{\xi}\,i\bar{\sigma}^\mu(\partial_\mu\xi) - W^\dagger_\phi W_\phi \\ &\quad -\frac{1}{2}\,(W_{\phi\phi}\,\xi\xi + W^\dagger_{\phi\phi}\bar{\xi}\bar{\xi})\end{aligned}$$

and the real potential is

$$V = W^\dagger_\phi W_\phi = m^2\phi^\dagger\phi + \frac{mh}{2}\,(\phi^\dagger\phi^2 + \phi^{\dagger 2}\phi) + \frac{h^2}{4}\,(\phi^\dagger\phi)^2$$

Exercise: If $\Psi = \begin{pmatrix}\xi \\ \bar{\xi}\end{pmatrix}$ is a Majorana 4-spinor, show that the Ψ-dependent part of the WZ Lagrangian can be written down as

$$\begin{aligned}\mathcal{L}_\Psi &= \frac{1}{2}\overline{\Psi}\,i\gamma^\mu\partial_\mu\Psi - \frac{1}{2}m\,\overline{\Psi}\,\Psi \\ &\quad -\frac{h}{2}\,\phi\,\overline{\Psi}\,P_L\,\Psi - \frac{h}{2}\,\phi^\dagger\overline{\Psi}\,P_R\,\Psi\,,\end{aligned}$$

where $P_{L,R} = (\mathbf{1}_4 \pm \gamma_5)/2$ and $\gamma_5 = {\rm diag}\,(\mathbf{1}_2\,,\,-\mathbf{1}_2)$.

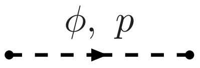
Summary

The complete WZ Lagrangian is

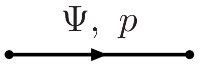
$$\begin{aligned}\mathcal{L}_{\rm WZ} &= (\partial^\mu\phi^\dagger)(\partial_\mu\phi) - m^2\phi^\dagger\phi + \frac{1}{2}\overline{\Psi}\,i\gamma^\mu\partial_\mu\Psi - \frac{1}{2}m\,\overline{\Psi}\,\Psi \\ &\quad -\frac{mh}{2}\,(\phi^\dagger\phi^2 + \phi^{\dagger 2}\phi) - \frac{h^2}{4}\,(\phi^\dagger\phi)^2 \\ &\quad -\frac{h}{2}\,\phi\,\overline{\Psi}\,P_L\,\Psi - \frac{h}{2}\,\phi^\dagger\overline{\Psi}\,P_R\,\Psi\,,\end{aligned}$$

where the F-field has been integrated out.

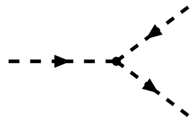
Feynman rules:



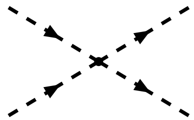
: $\dfrac{i}{p^2 - m^2}$



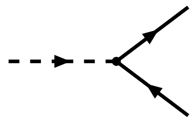
: $\dfrac{i}{\not{p} - m}$



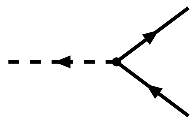
: $-imh$



: $-ih^2$



: $-ihP_L$



: $-ihP_R$

SUSY is such an elegant symmetry that it would be a pity if nature made no use of it!