Beam field and Space-charge effects

lecture

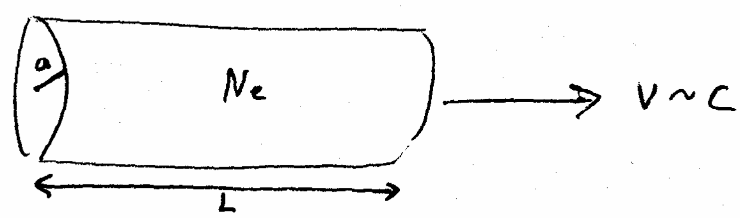
the fields in our dipole and quadrupole are DESIRED field

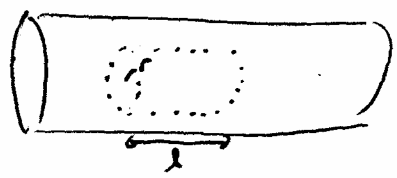
→ small diversion to look at undesired fields

→ one such field is the E.M. field of the beam itself, acting back on the beam → "Space-charge"

Consider cylindrical beam with uniform charge moving in $+Z$

N - particles
e = charge





Gaussian Surface
radius r, length l
(r can be greater than a)

• E field is radial along length (ignore end effects)
apply Gauss's law to Gaussian Surface

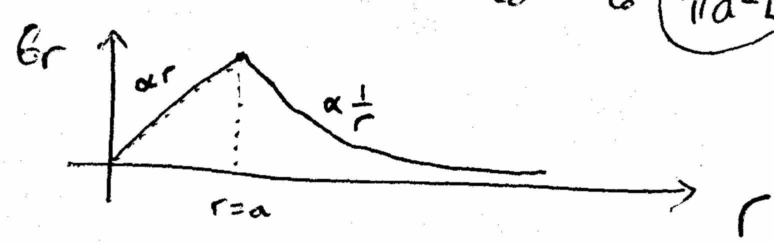
Inside Cylinder : $\oint E \cdot dA = \frac{q_{enc}}{\varepsilon_0}$

double integral

$$= E_r \cdot 2\pi r l = \frac{q}{\varepsilon_0} = \underbrace{\frac{Ne}{\pi a^2 L}}_{\text{charge density}} \cdot \underbrace{\pi r^2 l}_{\text{Volume of Surface}} \cdot \frac{1}{\varepsilon_0}$$

$$\Rightarrow E_r = \frac{Ne}{2\pi \varepsilon_0 a^2 l} \cdot r \qquad r < a$$

outside cylinder : $E_r \cdot 2\pi r l = \frac{q}{\varepsilon_0} = \frac{1}{\varepsilon_0} \underbrace{\frac{Ne}{\pi a^2 L}}_{\text{total charge density}} \underbrace{\pi a^2 l}_{\text{volume of surface}} \qquad r > a$



• to find the B field we apply Ampere's law $\oint \underline{B} \cdot \underline{ds} = I\mu_0$

charge enclosed

• inside beam $(r<a)$ charge enclosed by Gaussian Surface

$$q = \underbrace{\frac{Ne}{\pi a^2 L}}_{\text{charge density}} . \underbrace{\pi r^2 \ell}_{\text{volume of gaussian surface}}$$

"length" of charge ↓ veloci

this charge q goes past in time ℓ / V

$$\Rightarrow I_{enc} = \frac{q}{t} = \frac{qV}{\ell}$$

B is just B_θ around a current

$$B_\theta \oint ds = \mu_0 \frac{qV}{\ell}$$

$$\Rightarrow B_\theta \, 2\pi r = \frac{\mu_0 V Ne \pi r^2}{\pi a^2 L}$$

$r \to a$ on RHS

$$\Rightarrow B_\theta = \frac{\mu_0 V Ne}{2\pi a^2 L} \cdot r \qquad r<a$$

• outside the beam $B_\theta \, 2\pi r = \mu_0 \frac{qV}{\ell} = \frac{\mu_0 V Ne \pi a^2}{\pi a^2 L}$

$$\Rightarrow B_\theta = \frac{\mu_0 V Ne}{2\pi L} \frac{1}{r} \qquad r<a$$

<u>what is force experienced by beam particle inside the beam?</u>

we need fields inside beam, and $\underline{F} = e(\underline{E} + \underline{v} \times \underline{B})$

$$\underline{F} = e\underline{E} + e\,\underline{v} \times \underline{B}$$

$$= eE_r \hat{\underline{r}} + eV \hat{\underline{z}} \times B_\theta \hat{\underline{\theta}}$$

only have E_r, B_θ and z motion

$$= \frac{Ne^2}{2\pi \epsilon_0 a^2 L} r \hat{\underline{r}} + \frac{eV.\mu_0 V Ne}{2\pi a^2 L} . r \cdot \hat{\underline{z}} \times \hat{\underline{\theta}}$$

$(\hat{r}, \hat{\theta}, \hat{z})$ = R

$$= \frac{Ne^2}{2\pi \epsilon_0 a^2 L} r \hat{\underline{r}} - \frac{\mu_0 V^2 Ne^2}{2\pi a^2 L} r \hat{\underline{r}} \qquad \hat{\underline{z}} \times \hat{\underline{\theta}} = -\hat{\underline{r}}$$

$$= \frac{Ne^2}{2\pi \epsilon_0 a^2 L} r \hat{\underline{r}} \left(1 - \frac{V^2}{c^2}\right) \qquad \text{using } c^2 = \frac{1}{\epsilon_0 \mu_0}$$

2 terms: force from electric components and force from magnetic com[ponents]
- enter with opposite sign $\longrightarrow$ they tend to cancel
- perfect cancellation as $\gamma \longrightarrow \infty$ or $v \longrightarrow c$

($\frac{v}{c} \ll 1$ if low energy beam or heavy particles i.e ions or p[rotons]

this effect is called $\boxed{\text{Space-charge}}$ $\leftarrow$ low Energy, high mass (proton a[nd ions])

if Intensity is high $\rightarrow$ big effect. So S.C imposes a limit on the maximum intensity in an accelerator

[Notice: What if 2 beams collide with each other?
now, B field reverses direction as velocity of test particle reverses

"Beam-Beam interaction"
$\rightarrow$ now E and B fields ADD (pick up a "−" sig[n])
$\rightarrow$ Very Strong interaction
$\rightarrow$ Saved by force only acts in short time beams (

<u>Lets compare Lorentz forces and magnet forces experienced by a particle in an accelerator</u>

proton beam $a = 2mm$ $N = 10^{10}$ $L = 1m$ $\gamma = 2$ ($E = 1.8 GeV$

inside beam, we know $E_r = \frac{Ne}{2\pi \varepsilon_0 a^2 L} \cdot r$ $B_\theta = \frac{\mu_0 v N e}{2\pi a^2 L} \cdot r$

$$F^{Lorentz} = \frac{Ne^2}{2\pi \varepsilon_0 a^2 L \gamma^2} \cdot r\hat{\underline{r}}$$

forces and fields at $r = a$ are $E_r = 1.4 \times 10^4 \, V m^{-1}$
$B_\theta = 4 \times 10^{-5} \, T$
$F_r = 5.6 \times 10^{-16} \, N$

In a quadrupole magnet, with a pole tip at 5cm and a pole tip field of 1T, What is the force?

What do we mean?

Recall $\frac{1}{R(x,y,s)} = \frac{e}{p} B_z(x,y,s)$ (first lecture)

Expand $B_z(x)$ around nominal trajectory

$$B_z(x) = B_{z0} + \frac{\partial B_z}{\partial x} x + \frac{1}{2!}\frac{\partial^2 B_z}{\partial x^2} x^2 + \frac{1}{3!}\frac{\partial^3 B_z}{\partial x^3} x^3 + $$

typical expansion in accelerator physics

$$\frac{e}{p} B_z(x) = \frac{e}{p} B_{z0} + \frac{e}{p}\frac{\partial B_z}{\partial x} x + \frac{1}{2!}\frac{e}{p}\frac{\partial^2 B_z}{\partial x^2} x^2 + \cdots$$

$$= \frac{1}{R} + k x + \frac{1}{2!} m x^2 + \cdots$$

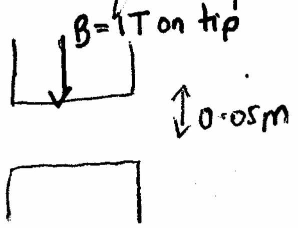
$k = \frac{e}{p}\frac{dB_z}{dx}$

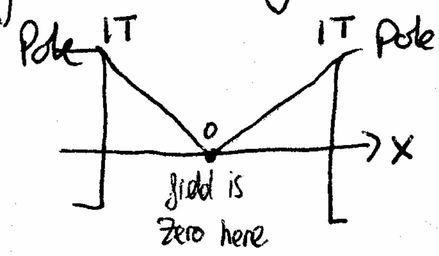
$m = \frac{e}{p}\frac{d^2B_z}{dx^2}$

dipole quadrupole sextupole

LINEAR optics (foundation of beam steering) NON-LINEAR optics

So, our quadrupole has a field increasing linearly in X





Q: how do you compute quadrupole k if you know g

$$\text{gradient} = g = \frac{1T}{0.05} = 20\ Tm^{-1} = \frac{\partial B_x}{\partial y} = \frac{\partial B_y}{\partial x}$$

$$B^{quad} = G(y\,\hat{\underline{x}} + x\,\hat{\underline{y}})$$

$\underline{F} = -evG\,y\,\underline{y} + evG\,x\,\underline{x}$

$|\underline{F}| = evGx$ (in x direction)

$\gamma = 2 \Rightarrow 2^2 = \frac{1}{1-v^2/c^2} \Rightarrow v = 0.25 \times 10^9\ ms^{-1}$ (note $v \neq c$)

x = edge of beam $= 0.002$ m

$e = 1.6 \times 10^{-19}\ J C^{-1}$

$|\underline{F}| = 1.7 \times 10^{-12}$ N due to quadrupole field

So, quadrupole force >> space-charge force — quad force only in magnet

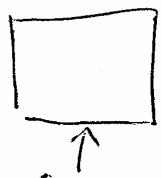
<u>but</u> space-charge force always acts, wherever beam goes so it can cause problems if γ is small

Relativistic Dynamics

why?

- essential to [illegible] particle behavior at high energies inside the accelerator
- "accelerator physics" - Physicist hat on.

Symmetry is abundant in nature eg. Snowflake, human being.



a Square

a Square is invariant under rotations of $n\pi/2$ + reflections $\left\langle \begin{array}{l} \text{diagonals} \\ \perp \text{ bisectors of the sides} \end{array} \right.$

"Group theory" is mathematical formulation

Other Symmetries — time invariance
translational invariance
Rotational invariance
Gauge invariance

eg. Newton's law of gravitation

$$m_1 \frac{d^2 \underline{x}_1}{dt^2} = G \frac{m_1 m_2 (\underline{x}_2 - \underline{x}_1)}{|(\underline{x}_2 - \underline{x}_1)|^3}$$

is invariant under translations (gravity same here and on Mars)
→ This form invariance is called "Covariance"

<u>Principle of Relativity</u>: "transformations between frames of that differ by constant velocity v are a fundamental symmetry of Nature"

Such frames are called <u>inertial frames</u>

<u>We can relate inertial frames using Lorentz Transformations</u>

~~frame~~ primed frame related to unprimed frame by a constant velocity in x_1 direction

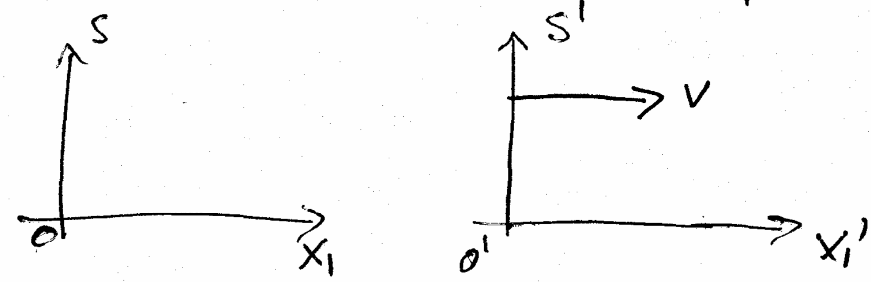
Then the 2 frames are related by

Will derive these later if time

$$\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$$

$$x_1' = \gamma(x_1 - vt) \quad (1) \qquad x_2' = x_2$$
$$t' = \gamma\left(t - \frac{vx_1}{c^2}\right) \quad (2) \qquad x_3' = x_3$$

Nothing more than a rotation (through an imaginary angle) in 4-dimensional space-time.



$$\left(\Rightarrow \ x_1 = \gamma(x_1' + vt') \quad (3) \qquad t = \gamma\left(t' + \frac{vx_1'}{c^2}\right) \quad (4) \right)$$

Consequence: 1) time dilation

clock at rest at O' $\Rightarrow x_1' = 0 \ \forall t'$

use (4) $\quad t = \gamma t'$

Since $\gamma > 1 \quad t > t'$ $\quad$ " moving clocks run slower "

2) length contraction

rod of length L along x_1' axis

$\Rightarrow$ ends at $x_1' = 0$ and $x_1' = L$

Consider measurement in S at fixed t ($t=0$)

use (1): $x_1' = \gamma x_1$

$\Rightarrow$ the ends are at 0 and $\frac{L}{\gamma}$ in S

so $\gamma > 1$ rod is contracted in S.

Note that $x_1^2 + x_2^2 + x_3^2 - c^2t^2$ is invariant under Lorentz transformations. Hence the quantity is called a Lorentz scalar (or a 4-scalar)

I shall not talk about velocity addition or relativistic Doppler effect, or light cones. (See a book!)

Energy and momentum transform just like space and time

$$p_1' = \gamma\left(p_1 - \frac{vE}{c^2}\right) \qquad p_2' = p_2$$

$$E' = \gamma(E - v p_1) \qquad p_3' = p_3$$

~~We can identify a Lorentz scalar in the same way~~

where we identify relativistic energy and momentum

$$E = \gamma m c^2 \qquad \underline{p} = \gamma m \underline{v}$$

We can identify a Lorentz scalar in the same way
$E^2 - c^2 p^2$ is our scalar $\equiv$ invariant mass

$$\Rightarrow \boxed{E^2 = p^2c^2 + m^2c^4}$$

relativistic energy-mom. relationship

Let's gain some insight:

$$\gamma(v) = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} \simeq 1 + \frac{v^2}{2c^2} + \cdots \qquad v \ll c$$

$$\Rightarrow \quad E = \gamma m c^2 \simeq \underbrace{mc^2}_{\text{rest energy}} + \underbrace{\tfrac{1}{2} m v^2}_{\text{classical kinetic energy}} + \cdots$$

So $E = \gamma m c^2$ is consistent with classical physics
AND Newton never spotted $E = mc^2$ as conservation of mass was implicit in his work
Einstein spotted mass and kinetic energy contribute to E

Similarly $\underline{p} = \gamma m \underline{v} \simeq m\underline{v}$ + corrections. Again, consistent

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4-Vectors

We want to build our theories using 4D objects → 4-vectors and 4-scalars. Why? they naturally contain the required symmetry and our theory will be automatically covariant. ~~under Lorentz transformations~~.

Spacetime 4-vector $\underset{\rightarrow}{X} = (\underline{x}, ict)$

$$\underset{\rightarrow}{X} \cdot \underset{\rightarrow}{X} = X_\mu X_\mu = \underline{x}^2 - c^2 t^2$$
$$= \text{Spacetime invariant}$$
$$= \text{4-scalar}$$

"take the norm"

[Similarly 4-velocity $\underset{\rightarrow}{u} = \frac{d\underset{\rightarrow}{X}}{d\tau}$ $\quad \tau =$ proper time

Not important to us! $\quad = \gamma(u)(\underline{u}, ic)$]

See a book

LT ≡ rotations in 4D spacetime (generalise $p = mv$)

Define: $\underline{p} = m\gamma\underline{v}$, $E = \gamma mc^2$

Energy-momentum 4-vector $P_\mu = m V_\mu = (\underline{p}, \frac{iE}{c})$

→ naturally conserved in isolated system

→ is invariant under LT

→ component by component $\sum_i \gamma(V_i) m_i \underline{V_i} = \text{constant}$

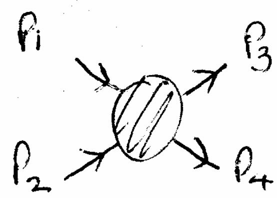
$$\sum_i \gamma(V_i) m_i c = \text{constant}$$

Note, we defined $E = \gamma mc^2$ and $\underline{p} = m\gamma\underline{v}$ as before

Relativistic kinematics

4-momentum conservation is a useful tool

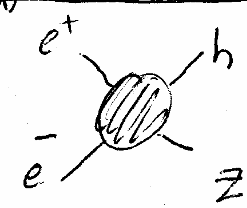
Consider 2→2 Scattering



$\Rightarrow P_1 + P_2 = P_3 + P_4$

<u>example - Higgs Boson Production at CERN (LEP)</u>

$e^+e^- \to hZ$



how? quantum electroweak dynamics
but, kinematics tell us only no Higgs produced above $115\ GeV/c^2$

at LEP, collide e^+e^- head on with energy $E_{beam} = 103\ GeV$

$$P_{e^+} = (\underline{p}, iE_{beam}/c) \qquad P_{e^-} = (-\underline{p}, iE_{beam}/c)$$

$$P_h = (\underline{p}', iE_h/c) \qquad P_Z = (-\underline{p}', iE_Z/c)$$

implicit use of conservation of momentum - LEP is in the "zero momentum" frame → h, Z produced back-to-back

$$2E_{beam} = E_h + E_Z \quad (1)$$

$$E_h^2 = c^2 p'^2 + m_h^2 c^4 \quad (2) \qquad E_Z^2 = c^2 p'^2 + M_Z^2 c^4 \quad (3)$$

for fixed E_{beam}, M_h is maximized if $p' = 0$

$$\Rightarrow E_h = M_h c^2 \qquad E_Z = M_Z c^2$$
$$\Rightarrow 2E_{beam} = M_h c^2 + M_Z c^2$$

we know $M_Z c^2 = 91\ GeV$ $\quad 2E_{beam} = 206\ GeV$

$$\Rightarrow M_h c^2 = 206 - 91 = 115\ GeV$$