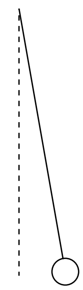
Lecture 2: Examples of SHM

1. Pendulum

A simple pendulum has a point mass m on the end of a light inextensible string of length L, which makes an angle θ with the vertical. The mass is acted on by gravity giving a force mg downwards. The mass does not move in the string direction so taking components along and perpendicular to the string means the net force is $mg\sin\theta \approx mgx/L$ in (nearly) a horizontal direction. (The other component gives the tension in the string, but we don't need to know that.)



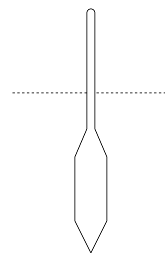
So the equation of motion is $m\ddot{x} = -mgx/L$

This is the SHM equation with k replaced by mg/L. So all the results of lecture 1 follow as the maths is the same, with

$\omega^2 = g/L \qquad T = 2\pi\sqrt{L/g}$

Notice that this doesn't depend on m.

2. A Floating object



Archimides' principle says the upward force on an object is the same as the weight of the water it displaces. Suppose a body has mass m and floats at a level with constant cross-section area A - for example a hydrometer with a cylindrical stem. If the level changes by x from equilibrium, the extra volume below (or above) the water level is Ax, and the force is $g\rho Ax$, where ρ is the density of water.

The equation of motion is $m\ddot{x} = -gA\rho x$ so there is simple harmonic motion with $\omega^2 = gA\rho/m$

3. Vertical Spring

Take the same spring as in Lecture 1 but make it more realistic by mounting it vertically, with the top fixed. The weight extends the spring.

The equations are now $F = m\ddot{x}$ and $F = -kx + mg$ This is not same as $F = -kx$. So why do we get SHM?

We can introduce a new variable $x' = x - mg/k$. The difference is just a constant so $\dot{x}' = \dot{x}$ and $\ddot{x}' = \ddot{x}$. The force is $F = -kx + mg = -k(x' + mg/k) + mg = -kx'$ so the equation of motion is $-kx' = m\ddot{x}'$

This is just the same as before, except for x' instead of x. It has the same SHM solutions (with the same $\omega = \sqrt{k/m}$).

The difference between x and x' is mg/k which is the amount by which the spring is extended by the weight. x' is just the deviation, measuring distances from the equilibrium point for the spring stretched by gravity, rather than the natural length of the spring.

4. Deforming a rod

Let's look in more detail at the spring constant k. Start with a simple linear rod (or wire).

Considering different pieces of the same material joined lengthways. The force is the same in all of them and it depends on the fractional increase - $F \propto x/L$

Now consider them joined sideways, and stretched by the same amount. The force is the same in all of them, but these forces add to give the total force. Force is proportional to area $F \propto A$

So considering various lengths and widths of the same material, we have $F \propto Ax/L$. Call this remaining proportionality constant Y (Young's modulus). It tells you how much the springiness of an object depends on the material itself, rather than the geometric size.

$F = -(YA/L)x$

It's also helpful to write $Y = (F/A)/(x/L) = stress/strain$

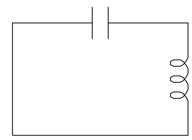
So for a particular rod (or wire) the spring constant is given by $k = YA/L$

Typical Y values are a few $\times 10^{10} N/m^2$. This seems a lot but area (for a wire, say) brings in factor of order 10^{-6} and strain is of order 10^{-3} so we get forces from a typical wire of a few tens of Newtons, which is reasonable

5. Electric circuit

Take a very basic circuit involving just an inductor of L Henrys and a capacitor of C Farads. Suppose something happens - for example - charging the capacitor externally or, waving a magnet past the inductor. There will be a current $I(t)$ through the circuit, which will vary with time.

The voltage across an Inductor is. $V = LdI/dt$
(that's the definition of inductance.)
The voltage across a Capacitance is $V = Q/C$ so $I = CdV/dt$
(that's the definition of capacitance.)
The total V =0 round the circuit (Kirchoff's Law) so



$$L\frac{dI}{dt} + \frac{Q}{C} = 0$$

Differentiate this again to turn the charge Q into the current I

$$L\frac{d^2I}{dt^2} + \frac{I}{C} = 0$$

rearranging gives

$$\frac{d^2I}{dt^2} = -\frac{1}{LC}I$$

I oscillates with angular frequency $\omega = \frac{1}{\sqrt{LC}}$

6. Torsion balance

In rotational problems we work with the angle θ, angular velocity $\dot{\theta}$ (or ω - confusion!) and angular acceleration $\ddot{\theta} \equiv \alpha$. The equivalent of Newton's second law is

$$\Gamma = I\ddot{\theta}$$

where Γ is the torque (in Nm) and I is the moment of inertia $\sum_i m_i r_i^2$, the distances being taken perpendicularly from the axis of rotation.

For a fibre or twisted spring: $\Gamma = -c\theta$
Putting them together : $\ddot{\theta} = -(c/I)\theta$ gives the SHM equation again, so it will oscillate with $\omega = \sqrt{\frac{c}{I}}$

7. Ubiquity of SHM

This wide variety of examples is not just coincidental. Suppose z is some completely general co-ordinate (it could be x, or V, or θ, or...). There will be a potential energy $U(z)$ which is some function of z. Left to itself the system will find its way to the z_0 for which $U(z)$ is minimum. The energy nearby can be written as a Taylor expansion:

$$U(z) = U_0 + (z - z_0)U' + \frac{1}{2}(z - z_0)^2U'' + ...$$

Make life simpler: take $z_0 = 0$ and $U_0 = 0$, and remember $U' = 0$ as this is a minimum. And neglect high order terms:

$$U(z) = \frac{1}{2}U''z^2$$

The (generalised) force is given by $F = -\frac{dU}{dz}$. Here this gives $F = -U''z$. The force is proportional to the displacement. So if you look at small oscillations about equilibrium for any system, you always get SHM.