Lecture 23: Matter Waves

1. Photons

Let's start with the photoelectric effect. Einstein (1905) said EM waves come in *quanta* of specific energy. $E = hf$ or $E = \hbar\omega$. EM Waves are particles, and the energy and frequency are linked.

Then relativity specifies $E^2 = p^2c^2 + m_0^2c^4$. For these packets $m_0 = 0$ so $E = pc$ or $p = E/c = hf/c = h/\lambda = \hbar k$

This is demonstrated experimentally through Compton Scattering. Photons of energy E (X rays or γ rays) scatter off electrons. The conservation laws for energy and momentum read

$$\vec{p} = \vec{p}\,' + \vec{p}_e \qquad E + m_ec^2 = E' + E_e$$

where E', p' refer to the scattered photon and E_e, p_e to the recoil electron. Notice the inclusion of the rest mass energy of the electron in the energy balance.

Take $\vec{p}\,'$ to the LHS, square the equation and multiply by c^2 to get

$E^2 + E'^2 - 2EE'Cos\theta = p_e^2c^2$

where θ is the scattering angle. The RHS is $E_e^2 - m^2c^4 = (E - E' + m_ec^2)^2 - m_e^2c^4 = E^2 + E'^2 - 2EE' + 2m_ec^2(E - E')$

Lots of cancelling gives $EE'(1 - cos\theta) = m_ec^2(E - E')$

and so $E' = \frac{m_ec^2E}{m_ec^2 + E(1 - cos\theta)}$.

This prescribes the energy of photons scattered at any angle. (In doing calculations it is useful to use $m_ec^2 = 511keV$.) It was verified slowly and with difficulty by Compton in 1922, nowadays it's part of 2nd year lab. This shows convincingly that $E = hf$ really means that we have to treat EM radiation as particle-like with momentum as well as energy.

Other waves are also quantised (e.g. phonons for sound) but don't qualify as particles.

2. The de Broglie relations

If waves behave like particles, maybe particles behave as waves. De Broglie suggested that particles (electrons, protons, billiard balls) have frequencies given by hf or $\hbar\omega$ and momenta given by h/λ or $\hbar k$. He picked out (correctly) the specific versions of the formulae to use from all of those listed above.

For a typical billiard ball, $p \approx 1kgm/s$ and $\lambda = h/p = 6\,10^{-34}$ which is ridiculously small.

For a typical electron, at $1m/s$ the momentum is $9\,10^{-31}kg\,m/s$ and the wavelength is of order 10^{-3} metres. At higher speeds the wavelength is of order the atomic size. So interference and diffraction may be noticeable and are seen (Davisson and Germer onwards.)

These waves have phase velocity $\omega/k = (\hbar\omega/\hbar k) = E/p$. If you write $E = \frac{1}{2}mv^2 = p^2/2m$ this gives $v_p = p/2m = v/2$ which is a bit of a surprise. If you write $E = m_0c^2 + ..$ you get $v_p = c^2/v$ which is even worse.

But the group velocity is $v_g = \frac{d\omega}{dk} = \frac{d(\hbar\omega)}{d(\hbar k)} = \frac{dE}{dp}$. With the classical formula, $\frac{dE}{dp} = (2p)/2m = v$. Relativistically $2EdE = 2pc^2dp$ so $\frac{dE}{dp} = pc^2/E = m_0\gamma vc^2/m_0\gamma c^2 = v$. Sanity is restored if you take the usual 'particle velocity' as being the group velocity.

3. From wave mechanics to quantum mechanics

These waves are of the form $\psi(x,t) = e^{i(kx - \omega t)}$. They are free waves/particles. How do we manage situations involving forces/potentials? An electron bound in a hydrogen atom is not going to be represented by an infinite plane wave.

Suppose such particles are described by some wave function $\psi(x,t)$. How do we find out about the energy and momentum of the particle from the function? We know that for a simple plane wave the answers to the energy question and the momentum question are $\hbar\omega$ and $\hbar k$. That suggests that the energy question is 'What do you get when you differentiate wrt t?' and the momentum question is 'What do you get when you differentiate wrt x?'

If we write $\hat{p} = -i\hbar\frac{\partial}{\partial x}$ and $\hat{E} = i\hbar\frac{\partial}{\partial t}$ then for plane waves we get the right answer

$$\hat{p}\psi = \hbar k\psi = p\psi \qquad \hat{E}\psi = \hbar\omega\psi = E\psi$$

so we suppose that this will be true in the more general case where forces and potentials are involved. There are many implications. It means that a 'measurement' can change a function. The differential of ψ may not be proportional to the original ψ. (In most literature, the energy question is called $\hat{H}$ rather than $\hat{E}$.)

The system is described by the energy - the dependence of the potential energy on position. Call this $V(x)$. Forces etc follow from that. We usually write down

$$Kinetic\ energy + potential\ energy = total\ energy$$

Because of this questioning business, we adapt that to

$$Kinetic\ energy \times \psi + potential\ energy \times \psi = total\ energy \times \psi$$

where for the KE we use $p^2/2m$ rather than mv^2 because we know how to find p: it's more basic than v.

$$\frac{\hat{p}^2}{2m}\psi + V(x)\psi = \hat{E}\psi$$

or

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + V(x)\psi(x,t) = i\hbar\frac{\partial\psi(x,t)}{\partial t}$$

which is the wave equation for matter - the Schrödinger equation.

Studying it would require a whole new lecture course.