Lecture 9: Waves

1. Waves on a string

Consider a piece of string with some tension T in it, and mass density μ kg/m.

It is distorted out of its natural straight-line configuration. Suppose the amplitude is some function f. This will depend on position and time so we have $f(x,t)$.

Consider an element δx and invoke Newton's second law. There is a force on it due to the tension T. Longitudinally this force balances. Transversely there is a force $T\frac{\partial f}{\partial x}$ at either end of the element. This almost cancels. It doesn't cancel exactly because of the difference between $\partial f/\partial x$ at x and $x+\delta x$. The net force is

$$T\frac{\partial f(x+\delta x)}{\partial x} - T\frac{\partial f(x)}{\partial x} \approx T\frac{\partial^2 f}{\partial x^2}\delta x$$

The ∂ symbol denotes partial differentiation: changing one variable while keeping the other constant.

The mass is $\mu\delta x$ and so ma is $\mu\frac{\partial^2 f}{\partial t^2}\delta x$. So Newton II says

$$T\frac{\partial^2 f}{\partial x^2} = \mu\frac{\partial^2 f}{\partial t^2}$$

Writing $v^2 = \frac{T}{\mu}$ this is

$$\frac{\partial^2 f}{\partial x^2} = \frac{\mu}{T}\frac{\partial^2 f}{\partial t^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} \qquad (1)$$

This is (in 1-D) The Wave Equation. The displacement f is a function of x and t, and the second derivatives are proportional to each other.

2. Longitudinal waves

Consider a solid bar, or a spring, of density μkg/m. One end is excited longitudinally (e.g. push-pull for a spring, hit it with a hammer if it's a bar). This motion is transmitted down the bar (or spring).

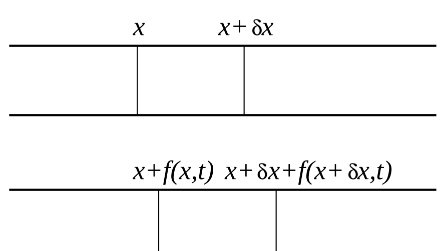
The disturbance is described by saying that at time t the point originally at x has moved by $f(x,t)$. f is the displacement, as before, but this time it's longitudinal instead of being transverse.

The portions of the bar are alternately expanded and compressed. This will produce a force T within the bar which is not constant but varies: it will be positive in the expansion regions and negative in the compression regions. The force will be proportional to the fractional increase (or decrease) in length: the strain S. Call the proportionality constant K

$$F = KS = K(\ell' - \ell)/\ell$$

For a spring with Hooke's law constant k and length L , $K = kL$. For a bar with Young's modulus $Y = (F/A)/S$, $F = AYS$ so $K = AY$

Consider a short length of the bar, originally from x to $x + \delta x$. It now goes from $x + f(x,t)$ to $x + \delta x + f(x + \delta x, t)$



Its length is $\delta x + f(x+\delta x,t) - f(x,t) \approx \delta x + \delta x\frac{\partial f}{\partial x}$ The strain is $\frac{\partial f}{\partial x}$ and the tension is $K\frac{\partial f}{\partial x}$.

What moves the bar is not the tension, but an imbalance of tension. Considering a short length δx, the force to the left is $K\frac{\partial f}{\partial x}$ evaluated at x. The force to the right is $K\frac{\partial f}{\partial x}$ evaluated at $x+\delta x$. The difference is $K\frac{\partial^2 f}{\partial x^2}\delta x$ The mass is $\mu\delta x$. Newton II gives

$$\mu\frac{\partial^2 f}{\partial t^2} = K\frac{\partial^2 f}{\partial x^2} \qquad (2)$$

Again, this is the wave equation, now with $v^2 = K/\mu$.

3. Solutions to the wave equation

The wave equation occurs in all sorts of physics as well as strings and springs, with various expressions for v.

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

This has an amazingly general solution. Any $f(x - vt)$ or $f(x + vt)$ is a solution.

Let $z = x - vt$. Then $f(x - vt)$ depends on z and only z. If there are different x and t values that give the same z, then f is the same.

If there is a change in x (with t held constant) there is an equal change in z and f changes by $\frac{df}{dz}\delta z = \frac{df}{dz}\delta x$ so $\frac{\partial f}{\partial x} = \frac{df}{dz} = f'(z)$. If there is a change in t then z changes by $-v\delta t$ and f changes by $-v\frac{df}{dz}\delta t$ so $\frac{\partial f}{\partial t} = -v\frac{df}{dz} = -vf'(z)$

This works again for the second differential.

$$\frac{\partial^2 f}{\partial x^2} = f''(z) \qquad (-v)^2 f''(z) = \frac{\partial^2 f}{\partial t^2}$$

so

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

and the same is true for $x + vt$.

4. Direction of travel.

Think about $f(x - vt)$. At some t it defines a waveform. Take a particular point on the waveform, corresponding to a particular value of $z = x - vt$. At some later time then to get the same value of z we have to increase x by an amount which is v times the increase in t. So as time goes by the waveform moves, with points on the waveform travelling in the $+x$ direction with speed v.

For $f(x + vt)$ the same argument can be gone through, except that as t increases you have to decrease x to preserve the same z. So $f(x + vt)$ represents a wave travelling in the $-x$ direction with speed v.

5. Sine Waves

Waves are usually discussed in terms of sines. Why? Because of Fourier Series: any $f(z)$ can be written as a sum of sines.

So let's look at $Asin[k(x \pm vt)]$

Considering the time dependance at some fixed x, this is just the Simple Harmonic Motion form, with amplitude A, a phase 'constant' kx, and an angular frequency $\omega = kv$. As before, one can use f or T instead of ω, according to convenience and/or taste.

Considering the x behaviour with fixed t, the $sin(kx)$ dependence can be alternatively expressed as $sin(2\pi x/\lambda)$, with $k \equiv \frac{2\pi}{\lambda}$. λ is the wavelength in metres. k is the wavenumber in m^{-1}

There is some confusion between frequency f and angular frequency $\omega = 2\pi f$ but it's not serious. There is TOTAL confusion between wavenumber $k = \frac{2\pi}{\lambda}$ and wavenumber $k = \frac{1}{\lambda}$ and it is serious.

We will use $k = \frac{2\pi}{\lambda}$ but other sources (including previous versions of this course) differ. You just have to be careful.

So again there is lots of choice: k or λ, ω or f or T. Sin or cos or complex exponential. At the two extremes lie: $Ae^{i(kx-\omega t)}$ and $Asin(2\pi(x/\lambda - t/T))$. As always, everything is right, it's a matter of using whichever formalism is most appropriate/helpful.

In any formalism, whatever multiplies t is v times whatever multiplies x

$$v = \lambda/T \qquad v = f\lambda \qquad v = \omega/k \tag{3}$$