Lecture 12: Music

1. Audible Frequencies

Human hearing spans the range from tens of Hz to a few kHz. The lowest audible frequency is around 16 Hz, and the piano goes down to 27 Hz. The human voice, for an exceptional bass singer, can get to 44 Hz, and a less exceptional one can manage 80 Hz. At the top end, a soprano's top D is at 2,300 Hz. The piccolo goes up to 3729 Hz, and the top note of a full-scale piano is at 4186 Hz.

'Concert' or 'musical' pitch defines the note A as 440 Hz. though historically it has varied from 370 to 567 Hz.'Scientific pitch' defines the note (middle) C as 256 Hz.

2. Notes in harmony

Pythagoras found that strings divided in simple ratios gave pleasing sounds. This is probably a basic feature of human psychology, not just cultural. The reason why is not understood. However we do now know that what matters is the frequencies of sounds, rather than the length directly.

2.1 The musical scale: the white notes

Take some fundamental frequency f. Call this the *tonic*. Say (for example) 256 Hz, and call the note C

Doubling the frequency to $2f$ or 512 Hz (which corresponds to halving the length of a string) gives a note which is so similar that we also call it C, the tonic, but one octave higher. ('Octave' is to be explained later.) Generally we identify any frequencies f, $2f$, $4f$, $f/2$... as being the same note.

The next-simplest numerical ratio is 3:2. Call $3/2f$ the *dominant* and in this case call the note G.

The next-simplest ratio is 4:3. $4/3f$ is called the *sub-dominant*, in this case the note F. (The name sub-dominant is because C is the dominant of F.)

Next comes 5:4. This is called the *mediant.* In this case the note is called E.

Let's pause there. We have 4 notes. The CEG chord is in the ratio 4:5:6 and sounds good (it's the 'C chord' on the guitar)

It would also be good to be able to play these chords for the dominant and subdominant. (G and F chords). For F we already have its dominant (C) and just need one new note: the mediant of the sub-dominant (which is also the subdominant of the mediant) and so has ratio $f \times 4/3 \times 5/4 = 5/3f$. Call it A. For a G chord we need the mediant of the dominant (also the dominant of the mediant) with frequency $f \times 3/2 \times 5/4 = 15/8f$, called B, and the dominant of the dominant, with frequency $f \times 3/2 \times 3/2 \rightarrow 9/8f$, called D

So we can assemble a table

Ratio	Name	Ratio in 48^{th}s	Interval	Note	Frequency	E-T frequency
1	tonic	48		C	256	256
3/2 x 3/2	dominant of dominant	54	1.125	D	288	287.3
5/4	mediant	60	1.111	E	320	322.5
4/3	subdominant	64	1.067	F	341.3	341.7
3/2	dominant	72	1.125	G	384	383.6
4/3 x 5/4	mediant of subdominant	80	1.111	A	426.6	430.5
3/2 x 5/4	mediant of dominant	90	1.125	B	480	483.3
2	octave	96	1.067	C	512	512

This is the just or natural scale or Zarlino's scale. It groups together a set of 7 different notes which will sound nice together. Putting them in sequence like this forms a scale of 8 notes (counting both C notes), hence the name 'octave'. As we started with the note C, this is called the 'key of C'

2.2 The keys of G and F: some black notes

In this table the first 4 columns are general. We could start with any note, any frequency, and establish the other 6 notes of the scale. So suppose that having established the key of C we go through the same procedure starting with the note G. The ratios all work as before, but starting with $3/2f = 384Hz$..

Of the 7 notes for the G scale we already have 5 from C: the notes G,B,D,C and E are included under both sets of ratios. The key of G does not use the notes F and A - the frequencies $4/3f$ and $5/3f$. But it does need its new mediant-of-dominant and dominant-of-dominant.

The first has frequency $5/4 \times 3/2 \times 3/2 = 45/32f$. This is 360 Hz and lies between F and G. Call it F$\sharp$.

The second has frequency $3/2 \times 3/2 \times 3/2f = 27/16f$. This is 81/48 or 432 Hz. This almost matches A, at 80/48 or 426.67 Hz. So we can identify this note with the old A and don't need to invent a new one.

So if you move to the key of G from the key of C, you have to play F$\sharp$ notes rather than F notes. This can be shown in the music by the key signature. You also have a problem with the note A: a skilled musician will play the note A differently, depending on whether they're in the key of C or the key of G. (If they can: with the voice or a violin or a trombone it is possible, though not with a trumpet or piano or guitar.)

If you start with F, the story is similar. F and its mediant and dominant are already in the scale. F's dominant is C, and its two partners are there. So we have 5 out of 7. We drop B and D.

F's subdominant has frequency 4/3 x 4/3 f or 85.33:48. That's less than B but more than A: it needs a new note - B♭. F's mediant-of-subdominant has frequency 53.33/48, almost matching D at 54, so we use it.

3. More black notes

So with two new notes (F♯ and B♭) we can play in three different keys, although 2 notes (A and D) aren't always quite right. This can be continued in both directions.

The scale built on the dominant of G (D) matches the notes in the scale of G except for one which is slightly wrong, and one which needs to be invented (C♯). Progressing to A, E, B each introduces a new sharp which is for the note below, so G♯, D♯, A♯. It also introduces a slight mismatch on one note - and these are cumulative.

The dominant of B is F♯. Its new note is E♯, though its frequency (15/16 x 3/2 x 3/2 x 3/2 x 3/2 x 3/2 x 3/2 $\rightarrow$ 64.07:48) matches pretty well onto the usual F of 64. Then C♯ gives B♯ which matches C. So for these 2 keys we don't need new notes on the keyboard, but we do have two fudges for each new key instead of one.

For the next 5 keys we again do not need new notes. G♯ gives F♯♯ which almost matches G. D♯ gives C♯♯ which almost matches D . A♯ needs a G♯♯ which almost matches A, and E♯ needs a D♯♯ which almost matches E. The dominant of E♯ is C, so we are back where we started.

The complete sequence is

Key	C	G	D	A	E	B	F♯	C♯	G♯	D♯	A♯	E♯	B♯=C
New note	-	F♯	C♯	G♯	D♯	A♯	(E♯)	(B♯)	(F♯♯)	(C♯♯)	(G♯♯)	(D♯♯)	-

We've covered 12 intervals of 3/2, which is a total of $1.5^{12} = 129.746...$ Which is almost 128, which is 7 octaves. The difference is known as 'the comma of Pythagoras'.

Continuing down the subdominants things happen the same way. At first, one new note is needed with every stage, and one is not quite right. After 5 flats have been introduced, no more are required but there are two imperfections as two flats and five double flats are matched to the basic notes.

The complete sequence is

Key	C	F	B♭	E♭	A♭	D♭	G♭	C♭	F♭	B♭♭	E♭♭	A♭♭	D♭♭=C
New note	-	B♭	E♭	A♭	D♭	G♭	(C♭)	(F♭)	(B♭♭)	(E♭♭)	(A♭♭)	(D♭♭)	-

4. Sharps and flats

We have 5 new sharps and 5 new flats. That's 10 new notes. Can we combine them?

Let's look at G♯ and A♭ (the third sharp to be introduced and the third flat). A G♯ has the frequency 15/8 times that of A, which has a frequency 3/2 that of D, which is 3/2 of G, which is 3/2 of C. Which totals to 405/64 or 6.328

An A♭ has a frequency 4/3 that of E♭ which is 4/3 of B♭ which is 4/3 of F which is 4/3 of C, which totals 256/81. Multiply by 2 to compare like with like gives 512/81 which is 6.321. So these are pretty close but not exactly. For an expert musician, A♭ and G♯ are different. Some keyboards have different keys (split black notes). But they're rare. Usually the corresponding sharp and flat are playable as the same note.

5. Consistency

In the early days this was fine. Some scales could be adjusted, others were just impossible - a primitive flute could only play in one or two keys. Then technology led to easier-to-play but less flexible instruments - the guitar has frets. Larger orchestras came in so tuning became more complicated. And keyboard instruments, especially the piano, can't be adjusted easily.

The solution used today is based on the fact that the intervals between notes in the scale are either 1.111 or 1.125 or 1.067. 1.111 is almost 1.125, and almost the square of 1.067. Call these intervals the tone and semitone. There are 12 semitones in the complete octave so make the semitone factor the twelfth root of 2. So the semitone ratio is $\sqrt[12]{2} = 1.05946$ - the tone is the square of this, 1.12246. If these ratios are used the frequencies are all a little bit wrong, but none of them is very wrong. This is shown in the right hand column of the original table

This equal-temperament scheme was pushed by J S Bach, who in his set of pieces 'The Well tempered Clavier' showed how all pieces written in different keys still sounded good with an equal-tempered system.