

# Lecture 1: Simple Harmonic Motion

## 1. Introduction

Vibrations and waves are everywhere in physics. Mechanics involves oscillating bodies: bridges and cranes and buildings. Optics is all about light waves. Music is all about sound waves and so, in a different way, is seismology. Solid State physics is all about the vibrations of atoms in crystal lattices, and the waves that result. General Relativity predicts gravitational waves. The fundamental particles (quarks, leptons, hadrons and so on) are waves, and every now and then remind us of their quantum nature.

## 2. The problem: a mass on a spring



The basic example of an oscillating system is a mass  $m$  on an ideal (i.e. massless and elastic) spring. The force is given by Hooke's Law

$$F = -kx \quad (1)$$

where  $x$  is the displacement from the equilibrium position and  $k$  is the *spring constant*. The response of the mass is given by Newton's 2nd law

$$F = ma \quad (2).$$

The acceleration can also be written as  $\ddot{x}$ , and putting the two equations together we have

$$m\ddot{x} = -kx$$

or

$$\ddot{x} = -\left(\frac{k}{m}\right)x$$

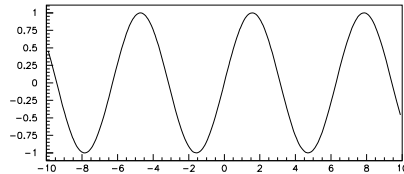
Writing, for future convenience,

$$\omega^2 = \frac{k}{m} \quad \text{or} \quad \omega = \sqrt{\frac{k}{m}} \quad (3)$$

this gives, writing it out in full

$$\frac{d^2x(t)}{dt^2} = -\omega^2x(t) \quad (4).$$

## 3. The Solution



Eq. 4 is a differential equation which governs how the position  $x$  of the mass varies with time. For the solution we need to find a function which, when differentiated twice, just gives the function you first thought of, multiplied by a minus sign and some constant. An obvious candidate is  $x(t) = \sin(\omega t)$  as differentiating  $\sin \rightarrow \cos \rightarrow -\sin$ . This is indeed a solution, but it's not the most general one: solving a second order differential equation will always introduce two arbitrary constants of integration. These are easy to spot as the amplitude and phase of the oscillation. If the function is multiplied by a constant  $A$  then Eq. 4 still holds, as the constant just multiplies both sides. So we have

$$x(t) = A\sin(\omega t)$$

If a phase  $\phi$  is added to the argument, the differentiation goes through as before. So the full solution is

$$x(t) = A\sin(\omega t + \phi) \quad (5)$$

The phase  $\phi$  can often be made to be zero by a suitable choice of the  $t = 0$  point.  $A$  is the amplitude of the oscillation. It has the same dimensions as  $x$  and the system oscillates between  $+A$  and  $-A$ . (So the peak-to-trough height is  $2A$ ).

What this says is: if you have a mass on the spring, then it's position will oscillate like a sine function, and the frequency of that oscillation is prescribed by the properties of the system:  $k$  and  $m$ , through Eq. 3. The amplitude and phase are not prescribed: they depend on how and when the system was excited.

#### 4. Velocity, Force and Energy

If the position at time  $t$  is  $A\sin(\omega t)$  then the velocity is  $v = \omega A\cos(\omega t)$  and the kinetic energy is  $KE = \frac{1}{2}mv^2 = \frac{1}{2}mA^2\omega^2\cos^2(\omega t)$

The force on the body is  $F(x) = -kx$  so the potential energy at position  $x$  is  $-\int_0^x F(x')dx' = \int_0^x kx'dx' = \frac{1}{2}kx^2$ . The Force is linear in the displacement so the potential energy is quadratic - that is the definition of 'Simple Harmonic Motion'. If  $x = A\sin(\omega t)$  then the Potential energy is  $PE = kA^2\sin^2(\omega t)$

Addition gives the total energy

$$E = KE + PE = \frac{1}{2}mA^2\omega^2\cos^2(\omega t) + \frac{1}{2}kA^2\sin^2(\omega t) = \frac{1}{2}kA^2$$

using the fact that  $m\omega^2 = k^2$  (Equation 3) and the familiar  $\sin^2 + \cos^2 = 1$

So if the amplitude is  $A$  the total energy is  $\frac{1}{2}kA^2$ , and this is converted from potential to kinetic energy and back again as the system oscillates.

#### 5. Frequency and time

$\omega$  was introduced in Eq. 3 as a piece of notation, It now emerges as the angular frequency of the oscillation. It is measured in radians per second. That is not always convenient and easy to apprehend: the radian isn't a very familiar unit and anyway these are not physically measurable angles.

It is sometimes more helpful to use the frequency  $f$  (also sometimes the Greek letter  $\nu$  is used), the number of cycles per second, or the period  $T$ , the number of seconds per cycle. A complete cycle is a meaningful quantity and the cycles per second used for  $f$  are also denoted as Hertz (Hz). These quantities are all related

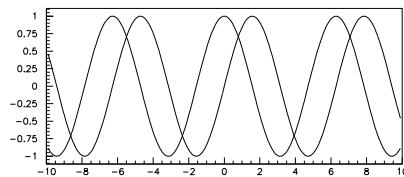
$$\omega = 2\pi f = \frac{2\pi}{T}$$

#### 6. Other forms of the solution

Equation 5 is the most general form. But there are other ways of writing it.

##### 6.1 Trigonometrical forms

What about  $\cos(\omega t)$ ? Or  $\sin(-\omega t)$ ? These both satisfy Eq. 4. But they are included in Eq. 5. You can see by algebra.  $\sin(\omega t + \frac{\pi}{2}) = \sin(\omega t)\cos\frac{\pi}{2} + \cos(\omega t)\sin\frac{\pi}{2} = \cos(\omega t)$ . Or graphically: the cosine curve is just the sine curve shifted by 90 degrees or  $\frac{\pi}{2}$



In principle you can use either form, but in practice you make life easy for yourself by sensible choice. If you have an oscillator displaced and released from rest at  $t = 0$  it's described more neatly by  $\cos(\omega t)$  than by  $\sin(\omega t + \frac{\pi}{2})$ . For an oscillator given an impulse at  $t = 0$  it is easier to use  $\sin(\omega t)$  rather than  $\cos(\omega t - \frac{\pi}{2})$ .

##### 6.2 The Complex form

Another function which satisfies Eq 4 is  $e^{i\omega t}$ . Differentiating twice just brings down a factor  $(i\omega)^2$  which is  $-\omega^2$ . This has the drawback that it's complex, while these displacements have got to be real. That's accounted for by just taking the real part. This is understood (and often not mentioned.) Again, there is an arbitrary amplitude and phase, the general solution is

$$x(t) = Ae^{i(\omega t + \phi)}$$

This is not a new solution, just another way of writing Eq 5. This can be seen through Euler's Theorem

$$e^{i\theta} = \cos\theta + i\sin\theta$$

So this is just  $A\cos(\omega t + \phi)$  again. The two constants  $A$  and  $\phi$  are sometimes encapsulated into the single complex constant  $\mathcal{A} \equiv Ae^{i\phi}$

Choosing which form to use is not a matter of finding the 'correct' one, just of suiting your own convenience.  $e^{i\omega t}$  is easier to handle in a complicated algebraic manipulation than  $\cos(\frac{2\pi t}{T})$ , but it's not so immediately easy to understand what it means.