Lecture 16: Group and Phase Velocity

Up to now we have just talked about the speed (or velocity) v of a wave. Actually there are two velocities: the Group Velocity v_g and the Phase Velocity v_p. They are given by the memorable formulæ:

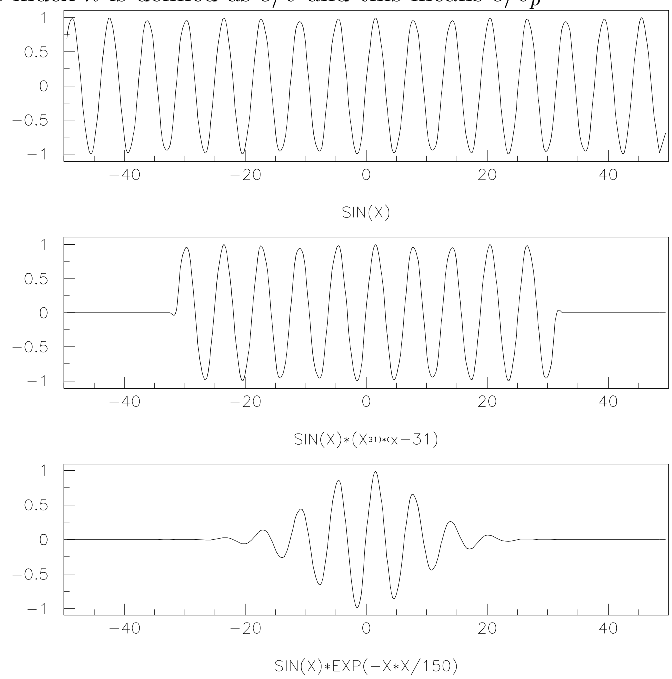
$$v_p = \frac{\omega}{k} \qquad v_g = \frac{d\omega}{dk}$$

1. Phase velocity

A single (infinite) wave is described by the expression $cos(\omega t - kx)$ or $sin[\frac{2\pi}{\lambda}(x - vt)]$ or equivalent.

The pattern travels with a velocity (actually a speed) $v_p = \frac{\lambda}{T} = f\lambda = \omega/k$

v_p is what matters with interference. The refractive index n is defined as c/v and this means c/v_p



2. Group velocity

An infinite wave is unrealistic.
A real wave has to have beginning and end.
The overall shape is called the *envelope*.
Various shapes are possible - abrupt or gentle.

$v_g = \frac{d\omega}{dk}$ is the velocity of the envelope.

3. Illustration

Consider two waves almost in step.
They have ω_1, k_1 and ω_2, k_2 (and λ_1, λ_2...)

Write the means and differences
$\omega = \frac{\omega_1+\omega_2}{2}, k = \frac{k_1+k_2}{2}$
$\Delta\omega = \frac{\omega_1-\omega_2}{2}, \Delta k = \frac{k_1-k_2}{2}$

the original quantities can be expressed in terms of these
$\omega_1 = \omega + \Delta\omega, \omega_2 = \omega - \Delta\omega$ etc

Adding the two waves gives a total wave
$e^{i(\omega_1 t - k_1 x)} + e^{i(\omega_2 t - k_2 x)}$

This can be written
$e^{i(\omega t + \Delta\omega t - kx - \Delta kx)} + e^{i(\omega t - \Delta\omega t - kx + \Delta kx)}$

Take out a common factor:
$e^{i(\omega t - kx)}\left(e^{i(\Delta\omega t - \Delta kx)} + e^{i(-\Delta\omega t + \Delta kx)}\right)$

Remembering $cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$ this is $2cos(\Delta\omega t - \Delta kx)e^{i(\omega t - kx)}$

The first term is clearly the envelope. It has small wavenumber and frequency and so a long wavelength and period. It travels with velocity $v_g = \Delta\omega/\Delta k$.

This generalises: $v_g = \frac{d\omega}{dk}$

4. Finding v_g

Often v_p is known from measurements or from basic principles. Take the expression for v_p and write ω/k for v_p in it. Turn all the λ and f etc terms into ω and k.

Then differentiate with respect to k. This gives an expression involving $\frac{d\omega}{dk}$ from which v_g can be extracted.

As a trivial example, suppose v_p is constant (i.e. independent of wavelength) with value c. Then $\omega = ck$ and $d\omega/dk = c$. Group and phase velocity are the same in this case.

5. Example: Refractive Index

The velocity of light in a medium tends to depend on the wavelength. (Hence rainbows, prisms, etc.). This is called *dispersion*. See the previous lecture for details.

People normally quote n as a function of λ rather than ω as a function of k. This contains the same information, we need to manipulate it: what follows is mere algebra:
$\frac{dn}{d\lambda} = \frac{dn}{dk}\frac{dk}{d\lambda}$
Take these two differentials separately.
First $n = \frac{c}{v_p} = \frac{ck}{\omega}$ so $\frac{dn}{dk} = \frac{c}{\omega} - \frac{ck}{\omega^2}\frac{d\omega}{dk}$

and secondly $k = \frac{2\pi}{\lambda}$ so $\frac{dk}{d\lambda} = -\frac{2\pi}{\lambda^2} = -\frac{k}{\lambda}$

Now put these together and get $\frac{dn}{d\lambda} = -\frac{k}{\lambda}(\frac{c}{\omega} - \frac{ck}{\omega^2}v_g)$

$\frac{dn}{d\lambda} = -\frac{kc}{\omega\lambda} + \frac{ck^2}{\omega^2\lambda}v_g = -\frac{c}{v_p\lambda} + \frac{c}{v_p^2\lambda}v_g$

Rearrange to get an expression for v_g

$v_g = \frac{v_p^2\lambda}{c}(\frac{c}{v_p\lambda} + \frac{dn}{d\lambda})$

$v_g = v_p + \frac{v_p\lambda}{n}\frac{dn}{d\lambda} = v_p(1 + \frac{\lambda}{n}\frac{dn}{d\lambda})$.

This can also be written: $v_g = v_p(1 - \frac{k}{n}\frac{dn}{dk}) = v_p - \lambda\frac{dv_p}{d\lambda} = \frac{c}{n+\omega(dn/d\omega)}$. Of all these equally-valid alternatives, only $v_g = \frac{d\omega}{dk}$ is *memorable.*

Note: if n is falling with λ, $v_g < v_p$. This is called *normal dispersion.* If n is rising, $v_g > v_p$ and this is called *anomalous dispersion*

6. Example

Suppose some glass has a slightly different refractive index for red light and blue light: $n = 1.51$ at 400 nm, $n = 1.49$ at 600 nm.

This is a fall of $\delta n = 0.02$ in $\delta\lambda = 200$ nm. The mean λ is 500 nm and the mean n is 1.5.

$v_p = 2 \times 10^8$ m/s $\qquad v_g = 2 \times 10^8\,(1 - (500/1.5)(.02/200)) = 2 \times 10^8(1 - .0333) = 1.93\,10^8$ m/s

7. Refractive index of X rays

We saw in the last lecture that at the highest frequencies(=shortest wavelengths, hence X rays) one can write $n = \sqrt{1 - B/\omega^2}$ where B is a positive number containing N and all the proportionality constants. This n is less than 1 so $v_p > c$ (!)

Putting that worry on one side, let's find the group velocity. Start from $n = \frac{c}{v_p} = \frac{ck}{\omega} = \sqrt{1 - \frac{B}{\omega^2}}$

Squaring: $\frac{c^2k^2}{\omega^2} = 1 - \frac{B}{\omega^2}$

Multiplying by ω^2: $c^2k^2 = \omega^2 - B$

Then differentiate wrt k: $2c^2k = 2\omega\frac{d\omega}{dk}$

This gives $v_g = c^2/v_p$

so $v_g < c$ and relativity is OK, as information (causal signals) travels with v_g not v_p.

In this case we have $v_g v_p = c^2$. The product of the group and phase velocities is equal to c^2. There are many cases where this turns out to be true, but it is not universal and there are some where it isn't.

8. A more general picture for deriving $v_g = \frac{d\omega}{dk}$

The earlier (standard) example just considered two waves. If you're happy with that, fine. For a more general approach we need to bring in an extended concept of Fourier Series.

A periodic function can be expressed as sum of sine and cosine terms

$f(\theta) = \sum_k a_k sin(k\theta) + b_k cos(k\theta) \qquad a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)sin(k\theta)d\theta \qquad b_k = \frac{1}{(2)\pi}\int_{-\pi}^{\pi} f(\theta)cos(k\theta)d\theta$

This can be extended to non-periodic functions

$f(x) = \int_{-\infty}^{\infty} F(k)e^{ikx}dk$ where $F(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx$

An infinite sine wave has a well-defined wavelength and thus a well-defined k. $F(k)$ is a delta function.

If $F(k)$ is broad, the wave is made up of lots of sine waves of different wavelengths and has a short wavepacket (as the contributions all cancel away from the peak).

Suppose we have a long wave packet $f(x)$. That means that there is a small spread in k. The function $F(k)$ will have a sharp peak about some central value k_0 At some initial time $t = 0$ the wave can be written

$f(x, 0) = \int_{-\infty}^{\infty} F(k)e^{ikx}dk$

After time t the wave has evolved to

$f(x, t) = \int_{-\infty}^{\infty} F(k)e^{i(kx - \omega(k)t)}dk$

where $\omega(k)$ explicitly shows that different wavelength components have different frequencies.

Let $\omega(k) = \omega_0 + \frac{d\omega}{dk}(k - k_0)$. This is the Taylor expansion to 1st order, and we're using the fact that the spread in k is small.

Then $f(x, t) = \int F(k)e^{i(k_0x + kx - k_0x - \omega_0t - (k - k_o)\frac{d\omega}{dk}t)}dk$

$f(x, t) = e^{i(k_0x - \omega_0t)}\int F(k)e^{i(k - k_0)(x - \frac{d\omega}{dk}t)}dk$

The first part is the pure sine wave. The second part (the integral) describes the envelope. It looks messy, but all the x and t dependence is in the $x - \frac{d\omega}{dk}t$. So the envelope progresses with velocity $\frac{d\omega}{dk} \equiv v_g$.