

Lecture 18: Two topics: Beats and The Doppler Effect

1. Beats

Suppose there are two waves whose angular frequencies are ω_1 and ω_2 . We'll consider sound waves as these are most familiar, though the phenomenon occurs for any sort of wave.

If these waves are (for simplicity) of equal intensity and in phase then the combined signal at some point x is $S(t) = \sin(\omega_1 t) + \sin(\omega_2 t)$

Using the famous trig formula $\sin A + \sin B = 2\sin(\frac{A+B}{2})\cos(\frac{A-B}{2})$ this is

$$S(t) = 2\sin(\frac{\omega_1 + \omega_2}{2}t)\cos(\frac{\omega_1 - \omega_2}{2}t) \quad (1)$$

This is general. Now suppose that the two frequencies are close: $\omega_1 \approx \omega_2$. Then the form of Eq 1 can be considered as an oscillation at the angular frequency $\frac{\omega_1 + \omega_2}{2} \approx \omega_1 \approx \omega_2$ modulated by an oscillation apparently at half the difference in frequency: $= \frac{\omega_1 - \omega_2}{2}$. But the 'beats' you hear are at twice that, as both $+1$ and -1 give maximally loud signals, and $\omega_{beat} = |\omega_1 - \omega_2|$.

The algebra works just as well in terms of the frequency f : $f_{beat} = f_1 - f_2$

This is readily observable with sounds. The closer the frequencies, the slower the beats.

For example: if two instruments, both tuned to middle C at 256 Hz, are playing the note E, but one plays it on the justified scale at 320 Hz and the other on the equal-tempered scale at 322.5 Hz, that causes the listener to hear (discordant!) beats at 2.5 Hz.

If the waves are not of equal intensity the beat effect is still there, it's just not so blatant.

2. Doppler. Moving Source and moving Observer

The Doppler effect - changes in frequencies when sources or observers are in motion - is familiar to anyone who has stood at the roadside and watched (and listened) to the cars go by. It applies to all types of wave, not just sound.

3. Moving Source

In time t the start of a wave emitted by the source travels ct . (c is the speed of the wave.) Suppose that the source travels with speed v (v is the speed of the source) in the same direction, so in time t the source travels vt . The distance between the head and tail is thus $(c - v)t$ and wavelength is contracted by a factor $(c - v)/c = 1 - v/c$

$$\lambda' = (1 - v/c)\lambda$$

This means (using $c = f\lambda$) that the frequency is higher

$$f' = c/\lambda' = \frac{1}{1 - v/c}f = \frac{c}{c - v}f \quad (2)$$

The same logic works for sources going away from you: for a source receding with speed v the frequency is lower

$$f' = \frac{1}{1 + v/c}f$$

In general the frequency is $f' = \frac{c}{c + v\cos\theta}f$ where θ is the appropriate angle.

This effect is appreciable. For example $v = 10$ m/s gives $f' = \frac{1}{1 - 10/344}f = 344/334f = 1.03f$ That's noticeable (remember that a semitone is a factor 1.059).

At 77 mph, which is 34.4 m/s, we have $v/c = 0.1$ and $f' = (1/0.9)f = 1.1111...f$.

4. Moving Observer

Now suppose the source is fixed but the observer is moving towards the source, with speed v . In time t , ct/λ waves pass a fixed point. A moving point adds another vt/λ . So

$$f' = (c + v)/\lambda = \frac{c + v}{c}f \quad (3)$$

At low speeds the difference between the two cases is small. For an observer moving at 77 mph the change is $f' = 1.1000f$, which is pretty close to the 1.111 we got for a moving source. $1/(1-v/c) \approx (1+v/c)$ if $v \ll c$.

But the $v = c$ limit is very different. For a moving source the limiting frequency is infinite, whereas for a moving observer the limiting frequency is $2f$.

5. Relativistic Doppler Effect

These results depend on the absolute velocities of the source and observer, not just on the relative velocity of the two. That seems odd, but is allowable as sound waves are waves in a medium, and motion relative to the medium may legitimately matter. But for light (or EM radiation in general) there is no medium, and this must be wrong. This needs relativity.

For a moving source and fixed observer, the factor $f' = \frac{1}{1-v/c}f$ still applies. However there is an extra factor as the source frequency is slowed down (time dilation) by the Gamma factor.

$$\text{So } f' = \frac{1}{\gamma} \frac{1}{1-v/c} f = \frac{1}{1-v/c} \sqrt{1-v^2/c^2} f = \frac{\sqrt{(1-v/c)(1+v/c)}}{(1-v/c)} f = \sqrt{\frac{1+v/c}{1-v/c}} f = \sqrt{\frac{c+v}{c-v}} f$$

If the source is regarded as fixed and the observer is moving, then the observer's clock runs slow. They will measure time intervals as being shorter than they are in the rest frame of the source, and so they will measure frequencies as being higher, again by a gamma factor.

$$f' = (1+v/c)/\sqrt{1-v^2/c^2} f = \sqrt{\frac{1+v/c}{1-v/c}} f \quad (4)$$

So for EM radiation, taking relativity into account, there is only one Doppler effect formula and it depends only on the relative velocity of source and observer.

6. Doppler effect and EM radiation.

In normal life, the Doppler effect for light is small as v/c is small. It is just about observable.

For example: 1000 m/s is a possible speed for a hot gas atom. $\frac{f'}{f} = \frac{1}{1-1000/3 \cdot 10^8} = \frac{300,000,000}{300,000,000-1000} = 1.000003$ - 3 parts per million. This is measurable by careful optical measurements ('Doppler Broadening').

A radar speed detector detects shifts in the reflected radar waves, which have a different frequency when reflected by a moving object. (The effect of reflected waves combines the moving source and object, with $\lambda' = \frac{c+v}{c-v}\lambda$, if you think about it.) This is small - vehicle speeds are of order 10^{-7} of the speed of EM radiation. These are measured by a cunning method called *heterodyning*. The reflected wave is combined with the original wave, and produces beats. The beat frequency is half the difference in the two frequencies. By using frequencies in the 100 MHz range, you get beat frequencies of order 10 Hz, which is measurable.

The Doppler effect is important in cosmology: receding galaxies (etc) have their lines shifted to longer wavelength ('red shift')